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# ST Decomposition Method for Solving Fully Fuzzy Linear Systems Using Gauss Jordan for Trapezoidal Fuzzy Matrices

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**Abstract**

In this article, we introduce ST decomposition procedure to solve fully fuzzy linear systems. This paper mainly to discuss a decomposition of non singular fuzzy matrix, Symmetric times triangular(ST) decomposition. Every non singular fuzzy matrix can be represented as a product of symmetric matrix S and triangular matrix T in the form of trapezoidal fuzzy number matrices. By this method, we obtain the solution of (either positive or negative) of fully fuzzy linear systems of equations in the case of trapezoidal fuzzy number matrices. A numerical example has been illustrated.

**Keywords:** Triangular decomposition, Symmetric Matrix, Trapezoidal fuzzy number matrices

## 1 Introduction

Zadeh introduced the concepts of fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to solve fuzzy metric spaces, fuzzy differential equations, fuzzy linear and non linear system etc. A major application of fuzzy number arithmetic is to solve fully fuzzy linear systems. Problems under Engineering, Physics and Economics should be represented by fuzzy rather than crisp numbers. We develop numerical procedures that would treat fully fuzzy linear system and solve them.

Solving fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy number vector was introduced by Friedman et al [6].

Some authors Mosleh [8] introduced LU decomposition method for solving fuzzy linear systems. M. Mosleh [8] introduced solving general fuzzy linear systems using ST decomposition procedure. Muzziloi et al. [9] developed fully fuzzy linear system of the form $A_1x + b_1 = A_2x + b_2$ where $A_1, A_2$ are square matrices of fuzzy coefficients, $b_1, b_2$ are fuzzy numbers. Dehgan et al. [2] considered fully fuzzy linear system of the form $Ax = b$ where $A$ is a fuzzy matrix, $x$ is a fuzzy vector, and the constant $b$ are vectors. Vijayalakshmi et al. [13] introduced the concept of solving fully fuzzy linear system for triangular fuzzy number matrices.

The structure of this paper is organized as follows. In section 2, preliminary concepts of trapezoidal fuzzy number matrices haave been discussed. In section 3, a new algorithm to solve fuzzy linear system in the form of trapezoidal fuzzy number matrices of order 4. In section 4 numerical example have been discussed to solve a solution using Gauss elimination method. In section 5, conclusion about the results have been discussed.

## 2 Preliminary Definitions

1. A fuzzy subset $\bar{A}$ of $R$ is defined by its membership function $\mu_{\bar{A}} : R \to [0, 1]$ which assigns a real number $\mu_{\bar{A}}$ in the interval [0,1] to each element $x \in R$ where the value of $\mu_{\bar{A}}$ shows the grade membership of $x$ in $\bar{A}$.

2. A trapezoidal fuzzy number denoted by $M =< m, n, \alpha, \beta >$ has the membership function

$$\mu_M(x) = \begin{cases} 0 & x \leq \alpha, \\ \dfrac{x-\alpha}{m-\alpha} & \alpha \leq x \leq m \\ 1 & m \leq x \leq n \\ \dfrac{\beta - x}{\beta - n} & n \leq x \leq \beta \\ 0 & x \geq \beta \end{cases}$$

Many authors have tried to define the basic operations on TRFNs. Here we introduce the definition of arithmetic operations due to Dubois and Prade [4].
Let $M =< m, n, \alpha, \beta >$ and $N =< x, y, \gamma, \delta >$ be two TRFNs

(a) **Addition:** $M + N =< m + x, n + y, \alpha + \gamma, \beta + \delta >$

(b) **Scalar Multiplication:** Let $\lambda$ be scalar then $\lambda M =< \lambda m, \lambda n, \lambda\alpha, \lambda\beta >$ when $\lambda \geq 0$, $\lambda M =< \lambda m, \lambda n, -\lambda\beta, -\lambda\alpha >$ when $\lambda \leq 0$. In particular $-M =< -m, -n, \beta, \alpha >$.

(c) **Subtraction:** $M - N =< m, n, \alpha, \beta > - < x, y, \gamma, \delta >=< m - x, n - y, \alpha + \delta, \beta + \gamma >$. For any 2 TRFNs $M$ and $N$ their addition, subtraction, and scalar multiplication $M + N$, $M - N$, $\lambda M$ are all TRFNs.

(d) **Multiplication:**
**Case 1** When $M \geq 0$, $N \geq 0$
$M.N =< m, n, \alpha, \beta > . < x, y, \gamma, \delta >$
$=< mx, ny, my + x\alpha, n\delta + y\beta >$
**Case 2** When $M \leq 0$, $N \leq 0$
$M.N =< mx, ny, x\alpha - m\delta, y\beta - n\gamma >$
**Case 3** When $M \leq 0$, $N \leq 0$
$M.N =< mx, ny, -m\gamma - x\alpha, -y\beta - n\delta >$

(e) **Exponentiation:** Using the definition of multiplication it can be shown that $M^n$ is given by
$M^n =< m, n, \alpha, \beta >^n \cong < m^n, n^n, -nm^{n-1}\beta, -nn^{n-1}\alpha >$ when $n$ is negative
$< m, n, \alpha, \beta >^n \cong < m^n, n^n, nm^{n-1}\beta, -nn^{n-1}\alpha >$ when $n$ is positive.

3. A matrix $\bar{A} = (a_{ij})$ is called a fuzzy matrix if each element of $\bar{A}$ is a fuzzy number. A fuzzy matrix $\bar{A}$ will be positive denoted by $\bar{A} > 0$ if each element of $\bar{A}$ be positive. To represent $n \times n$ fuzzy matrix $\bar{A} = (a_{ij})_{n\times n}$ such that matrix $a_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})$ with the new notation $\bar{A} = (A, B, M, N)$ where $A = (a_{ij})$, $B = (b_{ij})$, $M = (\alpha_{ij})$, $N = (\beta_{ij})$ are four $n \times n$ crisp matrices.

4. Consider the $n \times n$ fuzzy linear systems of equations

$$\begin{aligned}
(\bar{a}_{11} \otimes \bar{x}_1) \oplus (\bar{a}_{12} \otimes \bar{x}_2) \oplus \ldots (\bar{a}_{1n} \otimes \bar{x}_n) &= \bar{b}_1 \\
(\bar{a}_{21} \otimes \bar{x}_1) \oplus (\bar{a}_{22} \otimes \bar{x}_2) \oplus \ldots (\bar{a}_{2n} \otimes \bar{x}_n) &= \bar{b}_2 \\
&\cdots\cdots \\
(\bar{a}_{n1} \otimes \bar{x}_1) \oplus (\bar{a}_{n2} \otimes \bar{x}_2) \oplus \ldots (\bar{a}_{nn} \otimes \bar{x}_n) &= \bar{b}_n
\end{aligned}$$

The matrix of the above equation is $\bar{A} \otimes \bar{x} = \bar{b}$ where the coefficient matrix $\bar{A} = (a_{ij})$, $1 \leq i, j \leq n$ is a $n \times n$ fuzzy matrix and $\bar{x}, \bar{b} \in F(R)$. This system is called fully fuzzy linear system (FFLS).

**Proposed Method**

In this method, for solving the crisp linear system of equation $Ax = b$ is reduced to diagonal matrix by gauss elimination method and applying the back substitution to get the corresponding unknown values in the form of trapezoidal fuzzy numbers. Given any fuzzy linear system of equations in the form of trapezoidal fuzzy matrices that can be decomposed into the form such that $A = ST$ decomposition method whereas $S$ is the symmetric matrix and $T$ is upper triangular matrix. An algorithm has been introduced to rewritten A as the product of symmetric matrix and upper triangular matrix.

# 3 An Algorithm for Solving Trapezoidal Fuzzy Number Matrices of order 4 by ST Decomposition Method

Step I: Consider the non singular trapezoidal fuzzy number matrices $A$.

Set $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$

Step II: Decompose the matrix $A$, $A = ST$ where $S$ is symmetric matrix and $T$ is upper triangular matrix.

$$S = \begin{pmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{21} & s_{22} & s_{23} & s_{24} \\ a_{31} & s_{32} & s_{33} & s_{34} \\ a_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & t_{12} & t_{13} & t_{14} \\ 0 & 1 & t_{23} & t_{24} \\ 0 & 0 & 1 & t_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $t_{12} = \dfrac{a_{12} - a_{21}}{a_{11}}$, $\quad s_{22} = \dfrac{a_{11}a_{22} - a_{12}a_{21} + a_{21}^2}{a_{11}}$

$$s_{23} = \frac{a_{32}a_{11} - a_{31}a_{12} + a_{21}a_{31}}{a_{11}}, \quad s_{24} = \frac{a_{42}a_{11} - a_{41}a_{12} + a_{41}a_{21}}{a_{11}}$$

$$t_{23} = \frac{(a_{13} - a_{31})a_{21} - (a_{23} - s_{23})a_{11}}{a_{12}a_{21} - a_{11}a_{22}},$$

$$t_{13}=\frac{\begin{array}{l}a_{13}a_{21}a_{12}-a_{13}a_{11}a_{22}-a_{31}a_{21}a_{12}+a_{31}a_{11}a_{22}\\-a_{13}a_{21}^2+a_{31}a_{21}^2+a_{23}a_{11}a_{21}-s_{23}a_{11}a_{21}\end{array}}{a_{11}a_{12}a_{21}-a_{11}^2a_{22}}$$

$$s_{33}=\frac{\begin{array}{l}a_{33}a_{11}(a_{21}a_{12}-a_{11}a_{22})+a_{31}a_{13}a_{11}a_{22}-a_{31}^2a_{11}a_{22}-a_{32}a_{11}a_{13}a_{21}\\+a_{32}a_{11}a_{31}a_{21}+a_{32}a_{23}a_{11}^2-a_{32}s_{23}a_{11}^2-a_{31}a_{12}a_{23}a_{11}+a_{31}a_{12}a_{23}a_{11}\end{array}}{a_{11}(a_{12}a_{21}-a_{11}a_{22})}$$

$$t_{34}=\frac{\begin{array}{l}(a_{21}a_{31}-a_{11}s_{23})(a_{21}a_{14}-a_{21}a_{41}-a_{11}a_{24}+a_{11}s_{24})\\-(a_{21}^2-a_{11}s_{22})(a_{31}a_{14}-a_{31}a_{41}-a_{11}a_{34}+a_{11}s_{34})\end{array}}{(a_{21}a_{31}-a_{11}s_{23})^2-(a_{31}^2-a_{11}s_{33})(a_{21}^2-a_{11}s_{22})}$$

$$t_{24}=\frac{\begin{array}{l}(a_{31}^2-a_{11}s_{33})(a_{21}a_{14}-a_{21}a_{41}-a_{11}a_{24}+a_{11}s_{24})\\-(a_{21}a_{31}-a_{11}s_{23})(a_{31}a_{14}-a_{31}a_{41}-a_{11}a_{34}+a_{11}s_{34})\end{array}}{(a_{21}^2-a_{11}s_{33})-(a_{21}^2-a_{11}s_{22})(a_{21}a_{31}-a_{11}s_{23})^2}$$

$$t_{14}=\frac{\begin{array}{c}(a_{31}s_{23}-s_{33}a_{21})(s_{22}a_{14}-s_{22}a_{41}-a_{21}a_{24}+a_{21}s_{24})\\-(a_{31}s_{22}-a_{21}s_{23})(s_{23}a_{14}-s_{23}a_{41}-a_{21}a_{34}+a_{21}s_{34})\end{array}}{(a_{11}s_{22}-a_{21}^2)-(a_{31}s_{23}-s_{33}a_{21})(a_{31}s_{22}-a_{21}s_{23})(a_{11}s_{23}-a_{31}a_{21})}$$

$$s_{34}=\frac{\begin{array}{l}a_{43}a_{11}a_{21}a_{12}-a_{43}a_{11}^2a_{22}+a_{41}a_{13}a_{11}a_{22}-a_{41}a_{31}a_{11}a_{22}\\-a_{42}a_{11}a_{13}a_{21}+a_{42}a_{11}a_{31}a_{21}+a_{42}a_{23}a_{11}^2+a_{41}a_{11}a_{12}(a_{23}-s_{23})\end{array}}{a_{11}a_{21}a_{12}-a_{22}a_{11}^2}$$

$$s_{44}=a_{44}-a_{41}t_{14}-s_{24}t_{24}-s_{34}t_{34}$$

Step III: On solving $A\otimes x=b$ we have
$(A,B,M,N)\otimes(x,y,z,w)=(b,g,h,k)$
$Ax=b\Rightarrow x=A^{-1}b$
$By=g\Rightarrow y=B^{-1}g$
$Az+Mx=h\Rightarrow z=A^{-1}(h-Mx)$
$Bw+Ny=k\Rightarrow w=B^{-1}(k-Ny)$

Step IV: Replace $A=ST$, $B=S_1T_1$ we have $x=T^{-1}S^{-1}b$
$y=T_1^{-1}S_1^{-1}g$, $z=T^{-1}S^{-1}(h-Mx)$, $w=T_1^{-1}S_1^{-1}(k-Ny)$
using this formula we are finding the solution of $x,y,z,w$ using ST decomposition.

# 4 Numerical Example

Consider the following fully fuzzy linear systems in the form of 4×4 trapezoidal fuzzy matrices.

$(1, 2, 3, 5)\bar{x}_1 + (2, 3, 4, 5)\bar{x}_2 + (5, 6, 7, 8)\bar{x}_3 + (6, 7, 8, 9)\bar{x}_4 = (1, 2, 3, 4)$
$(3, 4, 7, 8)\bar{x}_1 + (4, 5, 6, 8)\bar{x}_2 + (8, 9, 10, 11)\bar{x}_3 + (7, 9, 11, 12)\bar{x}_4 = (2, 4, 6, 8)$
$(9, 10, 11, 12)\bar{x}_1 + (2, 4, 6, 7)\bar{x}_2 + (3, 4, 6, 7)\bar{x}_3 + (5, 7, 9, 11)\bar{x}_4 = (3, 5, 7, 11)$
$(4, 6, 7, 8)\bar{x}_1 + (1, 3, 4, 5)\bar{x}_2 + (2, 6, 7, 8)\bar{x}_3 + (6, 7, 8, 9)\bar{x}_4 = (4, 6, 8, 10)$

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 8 & 7 \\ 9 & 2 & 3 & 5 \\ 4 & 1 & 2 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 32 & 6 & 7 \\ 4 & 5 & 9 & 9 \\ 10 & 4 & 4 & 7 \\ 6 & 3 & 6 & 7 \end{pmatrix}, \quad \begin{pmatrix} 3 & 4 & 7 & 8 \\ 7 & 6 & 10 & 11 \\ 11 & 6 & 6 & 9 \\ 7 & 4 & 7 & 8 \end{pmatrix}$$

$$N = \begin{pmatrix} 5 & 5 & 8 & 9 \\ 8 & 8 & 11 & 12 \\ 12 & 7 & 7 & 11 \\ 8 & 5 & 8 & 9 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad g = \begin{pmatrix} 2 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \quad h = \begin{pmatrix} 3 \\ 6 \\ 7 \\ 8 \end{pmatrix}, \quad k = \begin{pmatrix} 4 \\ 8 \\ 11 \\ 10 \end{pmatrix}$$

Applying the above algorithm by ST decomposition we have $A = ST$ where

$$S = \begin{pmatrix} 1 & 3 & 9 & 4 \\ 3 & 7 & 11 & 5 \\ 9 & 11 & -33 & -27/2 \\ 4 & 5 & -27/2 & -173/56 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & -1 & 19/2 & 863/28 \\ 0 & 1 & -9/2 & -116/7 \\ 0 & 0 & 1 & 65/28 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$t_{12} = -1$, $s_{22} = 7$, $s_{23} = 11$, $s_{24} = 5$, $t_{23} = -9/2$, $s_{33} = -33$, $t_{34} = 65/28$, $s_{34} = -27/2$, $t_{24} = -110/3$, $t_{13} = 19/2$, $t_{14} = 863/28$, $s_{44} = -173/56$.

Applying Gauss Jordon method to find inverse,

$(A/I) = (I/A^{-1})$

$$S^{-1} = \begin{pmatrix} \frac{15003}{500} & \frac{-3336}{469} & \frac{2179}{1876} & \frac{-13}{67} \\ \frac{-3336}{469} & \frac{3837}{938} & \frac{-575}{938} & \frac{6}{67} \\ \frac{2179}{1876} & \frac{-575}{938} & \frac{303}{1876} & \frac{-13}{67} \\ \frac{-13}{67} & \frac{6}{67} & \frac{-13}{67} & \frac{28}{67} \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 1 & 1 & -5 & \frac{-1821}{56} \\ 0 & 1 & \frac{9}{2} & \frac{-585}{56} \\ 0 & 0 & 1 & \frac{-65}{28} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$x = T^{-1}S^{-1}b = \begin{pmatrix} \frac{-481}{10} \\ \frac{-6210}{469} \\ \frac{29}{67} \\ \frac{52}{67} \end{pmatrix}, \quad h - Mx = \begin{pmatrix} \frac{1528}{25} \\ \frac{2046}{5} \\ \frac{15149}{25} \\ \frac{48553}{125} \end{pmatrix}$$

$$Z = T^{-1}S^{-1}(h - Mx) = \begin{pmatrix} \frac{694209}{100} \\ \frac{53201}{100} \\ \frac{8826}{25} \\ \frac{6953}{100} \end{pmatrix}.$$

Consider $B = \begin{pmatrix} 2 & 3 & 6 & 7 \\ 4 & 5 & 9 & 9 \\ 10 & 4 & 4 & 7 \\ 6 & 3 & 6 & 7 \end{pmatrix}$ by above algorithm,

$$B = T_1S_1 \text{ where } T_1 = \begin{pmatrix} 1 & \frac{-1}{2} & 14 & \frac{647}{14} \\ 0 & 1 & -8 & \frac{-205}{7} \\ 0 & 0 & 1 & \frac{18}{7} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 2 & 4 & 10 & 6 \\ 4 & 7 & 9 & 6 \\ 10 & 9 & -64 & -30 \\ 6 & 6 & -30 & \frac{-122}{7} \end{pmatrix}$$

Finding the inverse using gauss Jordon method

$$S_1^{-1} = \begin{pmatrix} \frac{6943}{224} & \frac{-575}{28} & \frac{155}{112} & \frac{39}{32} \\ \frac{-575}{28} & \frac{96}{7} & \frac{-13}{14} & \frac{-3}{4} \\ \frac{155}{112} & \frac{-13}{14} & \frac{-1}{56} & \frac{3}{16} \\ \frac{39}{32} & \frac{-3}{4} & \frac{3}{16} & \frac{-7}{32} \end{pmatrix}, \quad T_1^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & -10 & \frac{-647}{14} \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$y = T_1^{-1} S_1^{-1} g = \begin{pmatrix} \frac{34151}{100} \\ \frac{225}{14} \\ \frac{89}{1000} \\ \frac{-15}{16} \end{pmatrix}$$

$$w = T_1^{-1} S_1^{-1}(k - Ny) \text{ where } K - Ny = \begin{pmatrix} \frac{-88809}{50} \\ \frac{-142119}{50} \\ \frac{-41999}{10} \\ \frac{-27947}{10} \end{pmatrix}$$

$$w = \begin{pmatrix} \frac{-837144}{5} \\ \frac{-382601}{50} \\ \frac{-3151}{5} \\ \frac{41203}{10} \end{pmatrix}$$

# Conclusion

In this article, a new methodology is applied to find the solution of fully fuzzy linear systems in the form of trapezoidal fuzzy matrices. We obtain both positive and negative solution of FFLS. An algorithm is introduced to solve trapezoidal fuzzy matrices by ST decomposition procedure method. Gauss Jordon method has been applied to find the corresponding inverses. We have discussed by taking an example of order 4 to solve fully fuzzy linear system of trapezoidal fuzzy matrices using ST decomposition method.

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