

Tense Operators on m -Symmetric Algebras

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Abstract

Here we initiate an investigation of the equational classes of m -symmetric algebras endowed with two tense operators. These varieties is a generalization of tense algebras. Our main interest is the duality theory for these classes of algebras. In order to do this, we require Urquart's duality for Ockham algebras and Goldblatt's duality for bounded distributive lattice with operations. The dualities enable us to describe the lattices of congruences on tense m -symmetric algebras.

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1 Introduction

In 1977, generalizing De Morgan algebras by omitting the polarity condition (i.e.: the law of double negation), J. Berman ([1]) began the study of which he called distributive lattices with an additional unary operation. Two years latter, A. Urquhart in [12] named them Ockham lattices with the justification that the so-called De Morgan laws are due, at least, in the case of propositional logic, to William of Ockham. These algebras are the algebraic counterpart of logics provided with a negation operator which satisfies De Morgan laws. Then recall that

An Ockham algebra is an algebra $\langle L, \wedge, \vee, N, 0, 1 \rangle$, where the reduct $\langle L, \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice and N is a unary operation satisfying the following conditions:

$$\begin{array}{ll} \text{(O1)} \ N(0) = 1, & \text{(O2)} \ N(1) = 0, \\ \text{(O3)} \ N(x \wedge y) = N(x) \vee N(y), & \text{(O4)} \ N(x \vee y) = N(x) \wedge N(y). \end{array}$$

The name Ockham algebras has become classical and from that moment on, many articles have been published about this class of algebras. Many of the results obtained have been reproduced in the important book by T. Blyth and J. Varlet ([2]), which may be consulted by any reader interested in broadening their knowledge on the topic.

For $m \geq 1$ and $n \geq 0$, $\mathcal{K}_{m,n}$ denotes the subvariety of Ockham algebras obtained by adjoining the equation $N^{2m+n}(x) = N^n(x)$. For $m \geq 1$ the algebras of the variety $\mathcal{K}_{m,0}$ are called m -symmetric algebras. (see [2], [13], [6])

The variety of 1-symmetric algebras is the variety of De Morgan algebras which contains Kleene algebras and Boolean algebras as subvarieties. Kleene algebras are obtained by adjoining the equation $(x \wedge N(x)) \vee (y \vee N(y)) = y \vee N(y)$, whereas Boolean algebras are obtained by adjoining the equation $x \wedge N(x) = 0$.

On the other hand, classical tense logic is a logical system obtained from bivalent logic by adding the tense operators G (*it is always going to be the case that*) and H (*has always been the case that*). It is well-known that tense algebras represent the algebraic basis for bivalent tense logic [3], [7].

Starting with other logical systems and adding appropriate tense operators, we produce new tense logics (see [5, 4]).

This paper deals with tense m -symmetric algebras, structures obtained from the m -symmetric algebras, by adding some tense operators. These algebras constitute a generalization of tense algebras. Our main interest is the duality theory for these classes of algebras. In order to do this, we require Urquhart's duality for Ockham algebras and Goldblatt's duality for bounded distributive lattice with operations. The dualities enable us to describe the

lattices of congruences on tense m -symmetric algebras.

2 Preliminaries

Even though the theory of Priestley spaces and its relation to bounded distributive lattices are well known (see [9], [10] and [11]), we shall recall some definitions and results with the purpose of fixing the notations used in this paper.

Recall that a Priestley space (or P -space) is a compact totally disconnected ordered topological space. If X is a P -space and $D(X)$ is the family of increasing, closed and open subsets of a P -space X , then $\langle D(X), \cap, \cup, \emptyset, X \rangle$ is a bounded distributive lattice.

On the other hand, let L be a bounded distributive lattice and $X(L)$ be the set of all prime filters of L . Then $X(L)$ ordered by set inclusion and with the topology having as a sub-basis the sets $\sigma_L(a) = \{P \in X(L) : a \in P\}$ and $X(L) \setminus \sigma_L(a)$ for each $a \in L$ is a P -space, and the mapping $\sigma_L : L \rightarrow D(X(L))$ is a lattice isomorphism. Besides, if X is a P -space, the mapping $\varepsilon_X : X \rightarrow X(D(X))$ defined by $\varepsilon_X(x) = \{U \in D(X) : x \in U\}$ is a homeomorphism and an order isomorphism.

If we denote by $\mathcal{L}$ the category of bounded distributive lattices and their corresponding homomorphisms and by $\mathcal{P}$ the category of P -spaces and the continuous increasing mappings (or P -functions), then there exists a duality between both categories by defining the contravariant functors $\Psi : \mathcal{P} \rightarrow \mathcal{L}$ and $\Phi : \mathcal{L} \rightarrow \mathcal{P}$ as follows:

- (P1) For each P -space X , $\Psi(X) = D(X)$ and for every P -function $f : X_1 \rightarrow X_2$, $\Psi(f)(U) = f^{-1}(U)$ for all $U \in D(X_2)$.
- (P2) For each bounded distributive lattice L , $\Phi(L) = X(L)$ and for every bounded lattice homomorphism $h : L_1 \rightarrow L_2$, $\Phi(h)(F) = h^{-1}(F)$ for all $F \in X(L_2)$.

On the other hand, H. Priestley ([9, 10, 11]), proved that if L is a bounded distributive lattice and Y is a closed subset of $X(L)$, then

- (P3) $\Theta(Y) = \{(a, b) \in L \times L : \sigma_L(a) \cap Y = \sigma_L(b) \cap Y\}$ is a congruence on L and that the correspondence $Y \mapsto \Theta(Y)$ is an anti-isomorphism from the lattice of all closed sets of $X(L)$ onto the lattice of all congruences on L .

The theory of Priestley topological duality was extended to Ockham algebras by A. Urquhart in [12]. This duality is equivalent to the one described now.

An Ockham space (or OP -space) is a pair (X, g) where X is a P -space and $g : X \rightarrow X$ is a decreasing continuous map. In addition, if (X_1, g_1) and (X_2, g_2) are OP -spaces, an OP -function from (X_1, g_1) to (X_2, g_2) is an increasing continuous function $f : X_1 \rightarrow X_2$ such that $f \circ g_1 = g_2 \circ f$.

Then, the following results are fulfilled:

- (U1) If (X, g) is an OP -space, then $(D(X), N_g)$ is an Ockham algebra where for all $U \in D(X)$, $N_g(U) = X \setminus g^{-1}(U)$,
- (U2) If (L, N) is an Ockham algebra, then $(X(L), g_N)$ is an OP -space where for all $P \in X(L)$, $g_N(P) = \{a \in L : N(a) \notin P\}$.

Moreover, these constructions give a categorical dual equivalence.

On the other hand, in [8] R. Goldblatt obtained a topological duality for bounded distributive lattices with operators, i.e. with a family of join-hemimorphisms and/or meet-hemimorphisms. Now, we will describe this duality in the particular case of bounded distributive lattices endowed with two unary meet-hemimorphisms, G, H , which from now on will be called G -lattices.

A gP -space is a triple (X, R_G, R_H) where X is a P -space, $R_G, R_H \subseteq X \times X$ are decreasing and the following conditions are satisfied:

- (R1) for every $x \in X$, $R_G^{-1}(x)$ and $R_H^{-1}(x)$ are closed subsets of X ,
- (R2) for each $U \in D(X)$, $G_{R_G}(U), H_{R_H}(U) \in D(X)$, where

$$G_{R_G}(U) = \{y \in X : R_G^{-1}(y) \subseteq U\}, H_{R_H}(U) = \{y \in X : R_H^{-1}(y) \subseteq U\}.$$

A gP -function from a gP -space (X_1, R_{G_1}, R_{H_1}) into another one, (X_2, R_{G_2}, R_{H_2}) , is a P -function $f : X_1 \rightarrow X_2$ which satisfies the following conditions:

- (r1) $(x, y) \in R_{G_1}$ implies $(f(x), f(y)) \in R_{G_2}$ for $x, y \in X_1$,
- (r2) $(y, f(z)) \in R_{G_2}$ implies that there is $x \in X_1$ such that $(x, z) \in R_{G_1}$ and $f(x) \leq y$ for $z \in X_1$ and $y \in X_2$,
- (r3) $(x, y) \in R_{H_1}$ implies $(f(x), f(y)) \in R_{H_2}$ for $x, y \in X_1$,
- (r4) $(y, f(z)) \in R_{H_2}$ implies that there is $x \in X_1$ such that $(x, z) \in R_{H_1}$ and $f(x) \leq y$ for $z \in X_1$ and $y \in X_2$.

In [8] it was shown that

- (G1) if (L, G, H) is an G -lattice and $R_T^L \subseteq X(L) \times X(L)$ is defined by $R_T^L = \{(P, F) \in X(L) \times X(L) : T^{-1}(F) \subseteq P\}$ for $T = G$ and $T = H$, then $(X(L), R_G^L, R_H^L)$ is a gP -space,

(G2) if (X, R_G, R_H) is a gP -space, then $(D(X), G_{R_G}, H_{R_H})$ is an G -lattice where G_{R_G} and H_{R_H} are defined in (R2).

Taking into account (G1) and (G2) it is proved that the category of gP -spaces and gP -functions is dually equivalent to the category of G -lattices and their corresponding homomorphisms.

3 Tense m -symmetric algebras

In this section, we will describe a topological duality for tense m -symmetric algebras bearing in mind the results indicated in Section 2 and [8].

Definition 3.1 *A tense m -symmetric algebra is an algebra $\langle L, \vee, \wedge, N, G, H, 0, 1 \rangle$ such that the reduct $\langle L, \vee, \wedge, N, 0, 1 \rangle$ is a bounded distributive lattice and N is a dual endomorphism satisfying the identity $N^{2m}(x) = x$ and G, H are unary operators on L verifying the following conditions:*

$$(T1) \quad G(1) = 1, \quad H(1) = 1,$$

$$(T2) \quad G(x \wedge y) = G(x) \wedge G(y), \quad H(x \wedge y) = H(x) \wedge H(y),$$

$$(T3) \quad x \leq G(N(H(N^{2m-1}(x))))), \quad x \leq H(N(G(N^{2m-1}(x)))).$$

In what follows, we will denote these algebras by (L, N, G, H) or simply by L where no confusion may arise.

Remark If $\langle L, \vee, \wedge, N, G, H, 0, 1 \rangle$ is a tense 1-symmetric algebra which satisfies the identity $x \wedge Nx = 0$, then $\langle L, \vee, \wedge, N, G, H, 0, 1 \rangle$ is a tense algebra.

By **TmS** we will denote the category of tense m -symmetric algebras and their corresponding homomorphisms.

Definition 3.2 *A tense m -symmetric space (or tms -space) is a system (X, g, R_G, R_H) where (X, g) is an OP -space, (X, R_G, R_H) is a gP -space and the following additional conditions are satisfied:*

$$(S1) \quad g^{2m}(x) = x,$$

$$(S2) \quad (x, y) \in R_G \text{ implies } (g(y), g(x)) \in R_H,$$

$$(S3) \quad (x, y) \in R_H \text{ implies } (g(y), g(x)) \in R_G.$$

A tms -function from a tms -space $(X_1, g_1, R_{G_1}, R_{H_1})$ into another one $(X_2, g_2, R_{G_2}, R_{H_2})$ is an OP -function $f : X_1 \rightarrow X_2$ which is also a gP -function.

Let **tmS** be the category whose objects are tms -space and whose morphisms are tms -functions.

Lemma 3.3 *If (X, g, R_G, R_H) is a tms-space, then $(D(X), N_g, G_{R_G}, H_{R_H})$ is a tense m -symmetric algebra, where for all $U \in D(X)$, $N_g(U)$, $G_{R_G}(U)$ and $H_{R_H}(U)$ are the sets defined in (U1) and (R2), respectively.*

Proof. We only prove (T3). Indeed, let $y \in U$ and $z \in R_G^{-1}(y)$. Suppose that $z \in g^{-1}(H_{R_H}(N_g^{2m-1}(U)))$. Then, $g(z) \in H_{R_H}(N_g^{2m-1}(U))$. From this last assertion, we infer that $R_H^{-1}(g(z)) \subseteq N_g^{2m-1}(U)$. Besides, since $(z, y) \in R_G$, then by virtue of (S2) we obtain that $g(y) \in R_H^{-1}(g(z))$ and so, $g(y) \in N_g^{2m-1}(U)$. Hence, $y \notin U$, which is a contradiction. Thus, $z \in N_g(H_{R_H}(N_g^{2m-1}(U)))$, from which we conclude that $R_G^{-1}(y) \subseteq N_g(H_{R_H}(N_g^{2m-1}(U)))$. So, $U \subseteq G_{R_G}(N_g(H_{R_H}(N_g^{2m-1}(U))))$. Similarly, it is proved that $U \subseteq H_{R_H}(N_g(G_{R_G}(N_g^{2m-1}(U))))$.

Lemma 3.4 *If (L, N, G, H) is a tense m -symmetric algebra, then $(X(L), g_N, R_G^L, R_H^L)$ is a tms-space, where g_N , R_G^L and R_H^L are those defined in (U2) and (G1), respectively.*

Proof. We only prove (S2). Indeed, let $a \in H^{-1}(g_N(P))$, and suppose that $a \notin g_N(F)$. Then $N(a) \in F$. Besides, from (T3) we have that $N(a) \leq G(N(H(a)))$. So, $G(N(H(a))) \in F$. On the other hand, from hypothesis we infer that $G^{-1}(F) \subseteq P$. Hence, $N(H(a)) \in P$. This last assertion allows us to conclude that $H(a) \notin g_N(P)$ which is a contradiction. Therefore, $a \in g_N(F)$ and so, $(g_N(F), g_N(P)) \in R_H^L$.

From Lemma 3.3 and 3.4, taking into account the results indicated in [12] and [8], we have Theorem 3.5.

Theorem 3.5 *The categories $\mathbf{TmS}$ and $\mathbf{tmS}$ are dually equivalent.*

This duality will allow us to determine the congruence lattice $Con_{tms}(L)$ of L .

Definition 3.6 *Let (X, g, R_G, R_H) be a tms-space. A closed subset Y of X is a tms-subset of X if it verifies these conditions for $u, v \in X$:*

(tms1) *if $v \in R_T^{-1}(u)$ and $u \in Y$, then there is $w \in Y$ such that $w \in R_T^{-1}(u)$ and $w \leq v$, for $T = G$ and $T = H$.*

(tms2) $Y = g^{2m-1}(Y)$.

We will denote by $\mathcal{C}_{tms}(X)$ the set of all tms-subset of X .

Lemma 3.7 *Let (L, N, G, H) be a tense m -symmetric algebra and $Y \in \mathcal{C}_{tms}(X(L))$. Then $\Theta(Y)$ is an tms-congruence, where $\Theta(Y)$ is defined as in (P3).*

Proof. We only prove that $\Theta(Y)$ preserves N , G and H . Let $(a, b) \in \Theta(Y)$. Since $Y = g^{2m-1}(Y)$, we have that $(Na, Nb) \in \Theta(Y)$. Indeed, suppose that $Q \in \sigma_L(N(a)) \cap Y$. Then, $N(a) \in Q$ and $Q \in g_N^{2m-1}(Y)$. This last assertion allows us to infer that there is $P \in Y$ such that $g_N^{2m-1}(P) = Q$. So, since $N(a) \in g_N^{2m-1}(P)$ we have that $a \notin P$. Hence, $b \notin P$ and therefore $N(b) \in Q$. The other inclusion is proved in a similar way.

On the other hand, let $F \in G_{R_G}(\sigma_L(a)) \cap Y$. Hence, $R_G^{-1}(F) \subseteq \sigma_L(a)$ and $F \in Y$. Suppose that $Q \in R_G^{-1}(F)$. Then, from (tms1) there is $W \in Y$ such that $W \subseteq Q$ and $W \in R_G^{-1}(F)$. This last assertion allows us to infer that $W \in \sigma_L(a)$, from which we conclude that $W \in \sigma_L(b) \cap Y$. So, since $W \subseteq Q$ we have that $Q \in \sigma_L(b)$. Hence, $F \in G_{R_G}(\sigma_L(b)) \cap Y$ and therefore, $G_{R_G}(\sigma_L(a)) \cap Y \subseteq G_{R_G}(\sigma_L(b)) \cap Y$. The other inclusion is proved in a similar way. Analogously, $\Theta(Y)$ preserves H .

Lemma 3.8 *Let (L, N, G, H) be a tense m -symmetric algebra, $\theta \in \text{Con}_{tms}(L)$ and $Y = \{\Phi(q)(F) : F \in X(L/\theta)\}$, where Φ is defined as in (P2). Then, $Y \in \mathcal{C}_{tms}(X(L))$.*

Proof. Let $\Phi(q) : X(L/\theta) \rightarrow X(L)$ be the function defined by $\Phi(q)(F) = q^{-1}(F)$ for $F \in X(L/\theta)$. Then, since $\text{Con}_{tms}(L)$ is a sublattice of $\text{Con}(L)$ we have that $Y = \{\Phi(q)(F) : F \in X(L/\theta)\}$ is a closed subset of $X(L)$ and $\theta = \Theta(Y)$. Besides, from [8],[12] we have that $\Phi(q)$ is a tms -function. In addition, Y is a tms -subset of $X(L)$. Indeed, let $V \in (R_G^L)^{-1}(U)$ with $U \in Y$. From this last assertion, there is $F \in X(L/\theta)$ such that $\Phi(q)(F) = U$. Then, $(V, \Phi(q)(F)) \in R_G^L$. So, from (r2) we have that there is $P \in X(L/\theta)$ such that $(P, F) \in R_G^{L/\theta}$ and $\Phi(q)(P) \subseteq V$. Therefore, from (r1) we infer that $\Phi(q)(P) \subseteq (R_G^L)^{-1}(U)$. Similarly, it is proved (tms1) for $T = H$.

On the other hand, let $Q \in Y$. Then, there is $F \in X(L/\theta)$ such that $\Phi(q)(F) = Q$. Since, $\Phi(q)$ is an OP -function we have that $g_N(Q) = \Phi(q)(g_N(F))$. This last assertion and (S1) allows us to conclude that $Q \in g_N^{2m-1}(Y)$. Therefore, $Y \subseteq g_N^{2m-1}(Y)$. The other inclusion is straightforward. This completes the proof.

From Lemma 3.7 and 3.8, we have Theorem 3.9.

Theorem 3.9 *Let (L, N, G, H) be a tense m -symmetric algebra and $(X(L), g_N, R_G^L, R_H^L)$ be the associated tms -space of L . Then, there is an anti-isomorphisms between $\text{Con}_{tms}(L)$ and the lattice $\mathcal{C}_{tms}(X(L))$.*

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