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Some Structural Properties of AG-Groups

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Abstract. An AG-group (an AG-groupoid with left identity and inverses) is a nonassociative structure in general midway between a quasigroup and an abelian group. This paper is meant to re-start the study of AG-groups. We prove that for an AG-group associativity and commutativity imply each other. Nonassociative AG-groups can never be power associative. The duality between left AG-groups and right AG-groups is also discussed.

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Preliminaries: A (*left*) *AG-groupoid* is a groupoid satisfying the *left invertive law:* $(ab)c = (cb)a$. This structure is also known as *left almost semigroup* (LA-semigroup) in [5], *left invertive groupoid* in [3], while *right modular groupoid* in [2, line 35]. An AG-groupoid $(S, \cdot)$ is called *locally associative AG-groupoid* if it satisfies $a\,(aa) = a\,(aa)$ for all $a \in S$. In the literature of loop theory *locally associativity* is called 3-PA (that is 3 *power associativity).* A *locally associative AG-groupoid* is *power associative. A right AG-groupoid or* a *right almost semigroup or shortly RA-semigroup (left modular groupoid in* [2, line 35]*)* is a groupoid satisfying the *right invertive law:* $a(bc) = c(ba)$. An *AG-groupoid* $(G, \cdot)$ is called a *(left) AG-group* or *a left almost group* (LA-group), if there exists left identity $e \in G$ (that is $ea = a$ for all $a \in G$), for all $a \in G$ there exists $a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = e$. Similarly a right *AG-groupoid* $(G, \cdot)$ is called *a right AG-group* or *a right almost group* (RA-group), if there exists right identity $e \in G$ (that is $ae = a$ for all $a \in G$), for all $a \in G$ there exists $a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = e$ see [6]. An element a of an AG-groupoid S is called left cancellative if $ax = ay \Rightarrow x = y$ for all

$x, y \in S$. Similarly an element a of an AG-groupoid S is called right cancellative if $xa = ya \Rightarrow x = y$ for all $x, y \in S$. An element a of an AG-groupoid S is called cancellative if it is both left and right cancellative. An AG-groupoid S is called left cancellative if every element of S is left cancellative. Similarly an AG-groupoid S is called right cancellative if every element of S right cancellative and it is called cancellative if every element of S is both left and right cancellative. Further study of the cancellativity of AG-groupoid can be found in [9].

Introduction: Kamran [4] extended the notion of AG-groupoid to AG-group. Later on the significant results on the topic were published in [6]. The present work provides a continuation of [6].

An AG-group is a generalization of abelian group and a special case of quasigroup. The structure of AG-group is a very interesting structure in which one has to play with brackets. There is no commutativity or associativity in general. But unlike groups and other structures, commutativity and associativity imply each other in AG-groups and thus AG-group becomes abelian group if any one of them is allowed (Theorem 1). The order of elements cannot be defined in AG-group. That is AG-group cannot be locally associative otherwise it becomes abelian group (Theorem 6). The duality between left AG-groups and right AG-groups has been shown in Theorem 3. To avoid excessive parenthesization, we will use the usual juxtaposition conventions , e.g., $ab \cdot c = (a \cdot b) \cdot c$. However for more clarity brackets will also be used.

Example 1. *Left AG-group of order* 3 :

$\cdot$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Example 2. *Right AG-group of order* 3 :

$\cdot$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Example 3. *Every abelian group* $(G, \cdot)$ *is an AG-group. This is the trivial case.*

Every abelian group$(G, \cdot)$ can be converted to an AG-group $(G, *)$ if $*$ is defined as $a * b = b \cdot a^{-1}$, by [4, Theorem 4.4, page 46].

Example 4. *The set* Z *of integers, is an AG-group under* $*$ *defined as* $a * b = b - a$ *see* [4, Example 4.2, page 45].

The only known examples of infinite AG-groups are of such type.

1. Basic Properties of AG-groups

The results of the following lemma seem to be at the heart of the theory of AG-groups; these facts will be used so frequently that normally we shall make no reference to this lemma.

Lemma 1. *The following conditions holds in AG-group G. Let $a, b, c, d, e \in G$, e is the left identity in G.*

(i) $(ab)(cd) = (ac)(bd)$ medial law, [2, Lemma 1.1(i)]
(ii) $ab = cd \Rightarrow ba = dc$, [7, Theorem 2.7, page 68]
(iii) $a \cdot bc = b \cdot ac$, [8, Lemma 4, page 58]
(iv) $(ab)(cd) = (db)(ca)$ paramedial law, [2, line 37]
(v) $(ab)(cd) = (dc)(ba)$, (we are unable to find reference)
(vi) $ab = cd \Rightarrow d^{-1}b = ca^{-1}$,
(vii) If e is right identity in G then it becomes left identity in G, i.e, $ae = a \Rightarrow ea = a$, [7, Theorem 2.3, page 67]
(viii) $ab = e \Rightarrow ba = e$,
(ix) $(ab)^{-1} = a^{-1}b^{-1}$[4, Remark 4.6, page 47].

Proof. (i) see [2, Lemma 1.1(i)]
(ii) $ba \cdot e = ea \cdot b = ab = cd = ec \cdot d = dc \cdot e \Rightarrow ba = dc$ since e is cancellative.
(iii) Since G satisfies $bc \cdot a = ac \cdot b$ so by (ii), we get $a \cdot bc = b \cdot ac$.
(iv) By applications of (i) and (ii), we have
$(ab)(cd) = (ac)(bd) \Rightarrow (cd)(ab) = (bd)(ac) \Rightarrow (cd)(ab) = (ba)(dc)$
$\Rightarrow (ab)(cd) = (dc)(ba) \Rightarrow (ab)(cd) = (db)(ca)$.
(v) By application of (iv) on the right hand side of (i).
(vi) $ab = cd \Rightarrow ab \cdot d^{-1} = c \Rightarrow d^{-1}b \cdot a = c \Rightarrow d^{-1}b = ca^{-1}$.
(vii) Using (iii) and then hypothesis $a = ae = a \cdot ee = e \cdot ae = ea$.
(viii) $ab = e = ee$ now apply (ii).
(ix)$(a^{-1}b^{-1})(ab) = (a^{-1}a)(b^{-1}b) = e \Rightarrow (ab)^{-1} = a^{-1}b^{-1}$. □

Most of the results of the above lemma are already proved in scattered papers on AG-groupoid given in the references. We only claim that our proofs are new and more standard.

The following lemma can make the calculations easier in AG-group.

Lemma 2. *Let G be an AG-group G and $a, b, c, d \in G$. Then the following hold.*

(i) $a(b \cdot cd) = a(c \cdot bd) = b(a \cdot cd) = b(c \cdot ad) = c(a \cdot bd) = c(b \cdot ad)$,
(ii) $a(bc \cdot d) = c(ba \cdot d)$,
(iii) $(a \cdot bc)d = (a \cdot dc)b$,
(iv) $(ab \cdot c)d = a(bc \cdot d)$.

Proof. (i) By repeated use of Lemma 1 part (iii).
(ii) Using Lemma 1 part (iii) and medial law, $a(b{\cdot}cd) = (bc)(ad) = (ba)(cd) = c(ba \cdot d)$.

(iii) Using Lemma 1 part (iii) and invertive law, $(a \cdot bc)d = (b \cdot ac)d = (d \cdot ac)b = (a \cdot dc)b$.

(iv) $(ab \cdot c)d = (dc)(ab) = a(dc \cdot b) = a(bc \cdot d)$. □

Theorem 1. *In an AG-groupoid with left identity and hence in an AG-group the following are equivalent.*

(i) Associativity.
(ii) Commutativity.

Proof. $(i) \Longrightarrow (ii)$ Suppose $(G, \cdot)$ is an AG-groupoid with left identity e. Let G be associative and $a, b \in G$. Then

$$\begin{aligned} ab &= e \cdot ab = ea \cdot b = ba \cdot e \\ &= b \cdot ae = (eb)(ae) = (ea)(be) \\ &= a \cdot be = ab \cdot e = eb \cdot a = ba. \end{aligned}$$

Thus G is commutative. $(ii) \Longrightarrow (i)$ is easy. □

Theorem 2. *An AG-group G with right identity e is abelian group.*

Proof. Let $a, b \in G$. Since $ab = ab \cdot e = eb \cdot a = ba$. Now apply Theorem 1. □

2. Duality between Left AG-groups and Right AG-groups

Here we prove that there is one-one correspondence between left AG-groups and right AG-groups. We shall prove that left AG-group and right AG-group are the opposite of each other.

Theorem 3. *Let $(G, \cdot)$ be a left AG-group. Define the operation $a * b = ba$ for every $a, b \in G$. Then $(G, *)$ is a right AG-group.*

Proof. Let $(G, \cdot)$ be a left AG-group. Define $a * b = ba$ for all $a, b \in (G, \cdot)$. Let $a, b, c \in G$.

$$\begin{aligned} a * (b * c) &= a * (cb) \\ &= (cb)a = (ab)c = c * (ab) = c * (b * a). \end{aligned}$$

Hence $(G, *)$ is a right AG-groupoid. If e is the left identity of $(G, \cdot)$ then $a * e = ea = a$. Hence e is the right identity of the right AG-groupoid $(G, *)$. It is clear that the inverses in $(G, *)$ remain the same as in $(G, \cdot)$. Hence $(G, *)$ is a right AG-group. □

From now onward we will consider only left AG-group and will call it simply AG-group.

3. Power Associativity of AG-groups

Lemma 3. *In an AG-groupoid S with the left identity e the following holds*

$$(ab)^2 = (ba)^2 \text{ for all } a, b \in S$$

Proof. Let $(S, \cdot)$ be an AG-groupoid S with the left identity e. Then for all $a, b \in S$. Then by using medial and paramedial laws, we have $(ab)^2 = (ab)(ab) = b^2a^2 = (ba)^2$. □

Theorem 4. *A locally associative AG-groupoid S with the left identity e in which every element is of order 2 is an abelian group.*

Proof. Let $(S, \cdot)$ be a locally associative AG-groupoid with the left identity e such that $a^2 = e$ for all $a \in S$.

If $a, b \in S$ then $a^2 = b^2 = e$. Also $ab \in S$ which implies that $(ab)^2 = e$ which further implies $(ba)^2 = e$ by Lemma 3. Now by using medial and paramedial laws and local associativity, we have

$$\begin{aligned} ab &= e(ab) = (ab)^2(ab) \\ &= (a^2b^2)(ab) = (a^2a)(b^2b) \\ &= (aa^2)(bb^2) = (b^2a^2)(ba) \\ &= (ba)^2(ba) = e(ba) = ba. \end{aligned}$$

Thus S is commutative. Hence S is commutative monoid. But every element of S is its own inverse and therefore S is an abelian group. □

Next we need the following theorem from [6] or [9].

Theorem 5. *An AG-group G is cancellative.*

Theorem 6. *An AG-group G with local associativity is abelian group.*

Proof. Let $(G, \cdot)$ be a locally associative AG-group. Then for all $a, b \in G$ by using medial and paramedial laws and local associativity of S, we have $(ab)(ab)^2 = (ab)(a^2b^2) = (aa^2)(bb^2) = (a^2a)(b^2b) = (ba)(b^2a^2)$
$= (ba)(ba)^2$, which by Lemma 3 and cancellativity implies $ab = ba$. Thus G is commutative. So the left identity becomes the right identity. Thus by Theorem 2, G is abelian group. □

Theorem 6 ensures that in a nonassociative AG-group orders of elements cannot be defined due to lack of local associativity. However we can speak of the order of an element up to 2.

Definition 1. An element a of order 2 of an AG-group G is called involution.

For example in Example 1 all elements are involutions. But this is not necessary that all elements of an AG-group must be involutions as the elements $4, 5, 6, 7$ are not involutions in the AG-group given in the following example.

Example 5. *An AG-group of order* 8.

$\cdot$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	3	0	1	2	6	7	5	4
2	2	3	0	1	5	4	7	6
3	1	2	3	0	7	6	4	5
4	6	4	7	5	2	0	1	3
5	7	5	6	4	0	2	3	1
6	4	7	5	6	3	1	2	0
7	5	6	4	7	1	3	0	2

4. General Properties of AG-groups

Theorem 7. *An AG-group G has exactly one idempotent element, which is the left identity.*

Proof. Suppose a is an arbitrary element of G such that $aa = a$. Then $ea = a = aa$ which implies $a = e$ by right cancellation. Thus the left identity e is the only idempotent element of G. □

Theorem 8. *A subset containing all the involutions of an AG-groupoid S with the left identity e is an AG-group contained in S.*

Proof. Let $(S, \cdot)$ be an AG-groupoid with the left identity e. Let $H = \{a \in S\colon a^2 = e\}$. H is non-empty since $ee = e \Rightarrow e \in H$. If $f, g \in H$ then $f^2 = g^2 = e$. Now $(fg)^2 = f^2g^2 = ee = e$. This implies $fg \in H$. Thus H is an AG-subgroupoid of S. Also since every element in H is its own inverse. Hence H is an AG-group. □

5. Future Directions

We have given some examples of AG-groups in this paper but we are unable to say how many AG-groups are there of certain order. So computational work (as one of the authors has done for IP loops (loops having inverse property) in [1] and [10]) is suggested because having enumeration of all AG-groups of order n will definitely give more insight of the structure of AG-groups. Moreover parallel study of AG-groups can be done with quasigroups, loops and groups.

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