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On Quadratic Diophantine Equation $x^2 - (t^2 - t)y^2 - (16t - 4)x + (16t^2 - 16t)y = 0$

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Abstract

Let $t \geq 2$ be a positive integer. Extending the work of A. Tekcan, here we consider the number of integer solutions of Diophantine equation $E\ :\ x^2 - (t^2 - t)y^2 - (16t - 4)x + (16t^2 - 16t)y = 0$. We also obtain some formulas and recurrence relations on the integer solution (x_n, y_n) of E.

Keywords: Pell's equation, Diophantine equation

1 Introduction

Let $t \geq 2$ be an integer. In [2], A. Tekcan consider the number of integer solutions of Diophantine equation $D\ :\ x^2-(t^2-t)y^2-(4t-2)x+(4t^2-4t)y = 0$ over $\mathbb{Z}$. He also derive some recurrence relations on the integer solutions (x_n, y_n) of D. In the present paper, we consider the integer of Diophantine equation

$$E\ :\ x^2 - (t^2 - t)y^2 - (16t - 4)x + (16t^2 - 16t)y = 0 \tag{1}$$

over $\mathbb{Z}$, where $t \geq 2$ be an integers. The reader can find many references in the subject in [1].

2 Resolution of The Diophantine Equation $x^2-(t^2 - t)y^2 - (16t - 4)x + (16t^2 - 16t)y = 0$

Note that the resolution of E in its present form is very difficult, that is, we can not determine how many solutions E has and what they are. So, we have to transform E into an appropriate Diophantine equation which can be easily solved. To get this let

$$T \; : \; \begin{cases} x = u + h \\ y = v + k \end{cases} \tag{2}$$

be a translation for some h and k.

By applying the transformation T to E, we get

$$\begin{array}{lll} T(E) & := \widetilde{E}: & (u+h)^2 - (t^2-t)(v+k)^2 - (16t-4)(u+h) \\ & & +(16t^2-16t)(v+k) = 0 \end{array} \tag{3}$$

In (3), we obtain $u(2h + 4 - 16t)$ and $v(-2kt^2 + 2kt + 16t^2 - 16t)$. So we get $h = 8t - 2$ and $k = 8$. Consequently for $x = u + 8t - 2$ and $y = v + 8$, we have the Diophantine equation

$$\widetilde{E} \; : \; u^2 - (t^2 - t)v^2 = 32t + 4 \tag{4}$$

which is a Pell equation.

3 Results

Now, we try to find all integer solutions (u_n, v_n) of $T(E)$ and then we can retransfer all results from $T(E)$ to E by using the inverse of T.

Theorem 3.1 *Let $\widetilde{E}$ be the Diophantine equation in (3), then*

(1) The fundamental solution of $\widetilde{E}$ is $(u_1, v_1) = (8t - 2, 8)$.

(2) Define the sequence (u_n, v_n) by

$$\begin{cases} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 8t-2 \\ 8 \end{pmatrix} \\ \\ \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 2t-1 & 2t^2-2t \\ 2 & 2t-1 \end{pmatrix}^{n-1} \begin{pmatrix} u_1 \\ v_2 \end{pmatrix}, \; \forall n \geq 2. \end{cases} \tag{5}$$

Then (u_n, v_n) is a solution of $\widetilde{E}$.

(3) The solutions (u_n, v_n) satisfy the recurrence relations

$$\begin{cases} u_n = (2t-1)u_{n-1} + (2t^2-2t)v_{n-1} \\ \\ v_n = 2u_{n-1} + (2t-1)v_{n-1} \end{cases} \tag{6}$$

for $n \geq 2$

(4) The solutions (u_n, v_n)

$$\begin{cases} u_n = (2t-1)u_{n-1} + (2t^2-2t)v_{n-1} \\ \\ v_n = 2u_{n-1} + (2t-1)v_{n-1} \end{cases} \tag{7}$$

for $n \geq 4$
(5) The n-th solution (u_n, v_n) can be given by

$$\frac{u_n}{v_n} = \left[t-1; \underbrace{2, 2t-2, \cdots, 2, 2t-2}_{n-1 \ times}, 1, 3 \right], \forall n \geq 1. \tag{8}$$

Proof.
(1) It is easily seen that $(u_1, v_1) = (8t-2, 8)$ is the fundamental solution of $\widetilde{E}$, since $(8t-2)^2 - 64(t^2-t) = 32t+4$.

(2) We prove it using the method of mathematical induction. Let $n = 1$, by (5) we get $(u_1, v_1) = (8t-2, 8)$ which is the fundamental solution and so is a solution of $\widetilde{E}$. Now, we assume that the Diophantine equation (4) is satisfied for n, that is $\widetilde{E} \ : \ u_n^2 - (t^2-t)v_n^2 = 32t+4$. We try to show that this equation is also satisfied for $n+1$. Applying (5), we find that

$$\begin{aligned} \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} &= \begin{pmatrix} 2t-1 & 2t^2-2t \\ 2 & 2t-1 \end{pmatrix}^n \begin{pmatrix} u_1 \\ v_2 \end{pmatrix} \\ &= \begin{pmatrix} 2t-1 & 2t^2-2t \\ 2 & 2t-1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \\ &= \begin{pmatrix} (2t-1)u_n + (2t^2-2t)v_n \\ 2u_n + (2t-1)v_n \end{pmatrix} \end{aligned} \tag{9}$$

Hence, we conclude that

$$\begin{aligned} u_{n+1}^2 - (t^2-t)v_{n+1}^2 &= \left[(2t-1)u_n + (2t^2-2t)v_n\right]^2 - (t^2-t)\left[2u_n + (2t-1)v_n\right]^2 \\ &= u_n^2 - (t^2-t)v_n^2 = 32t+4. \end{aligned}$$

So (u_{n+1}, v_{n+1}) is also solution of $\widetilde{E}$.

(3) Using (9), we find that

$$\begin{cases} u_n = (2t-1)u_{n-1} + (2t^2-2t)v_{n-1} \\ \\ v_n = 2u_{n-1} + (2t-1)v_{n-1} \end{cases}$$

for $n \geq 2$

(4) We prove it using the method of mathematical induction. For $n = 4$, we get
$u_1 = 8t - 2$
$u_2 = 32t^2 - 28t + 2$
$u_3 = -48t^2 + 72t - 2$
and
$u_4 = 512t^4 - 960t^3 + 544t^2 - 92t + 2$. Hence $u_4 = (4t-3)(u_3 + u_2) - u_1$. So $u_n = (4t-3)(u_{n-1} + u_{n-2}) - u_{n-3}$. is satisfied for $n = 4$. Let us assume that this relation is satisfied for n, that is,

$$u_n = (4t-3)(u_{n-1} + u_{n-2}) - u_{n-3}. \tag{10}$$

Then using (9) and (10), we conclude that

$$u_{n+1} = (4t-3)(u_n + u_{n-1}) - u_{n-2},$$

completing the proof.

Similarly, we prove that $v_n = (4t-3)(v_{n-1} + v_{n-2}) - v_{n-3},\ \forall n \geq 4$.

(5) We prove it using the method of mathematical induction. For $n = 1$, we have

$$\begin{aligned} \frac{u_1}{v_1} &= \frac{8t-2}{8} = t - 1 + \cfrac{1}{1 + \cfrac{1}{3}} \\ \\ &= [t-1; 1, 3] \end{aligned}$$

which is the fundamental solution of $\widetilde{E}$. Let us assume that the n-th solution (u_n, v_n) is given by

$$\frac{u_n}{v_n} = \left[t - 1; \underbrace{2, 2t-2, \cdots, 2, 2t-2}_{n-1 \ times}, 1, 3 \right].$$

and we show that it holds for (u_{n+1}, y_{n+1}).

Using (6) , we have

$$\begin{aligned}\frac{u_{n+1}}{v_{n+1}} &= \frac{(2t-1)u_n+(2t^2-2t)v_n}{2u_n+(2t-1)v_n}\\ &= \frac{2(t-1)u_n+u_n+(2t-1)(t-1)v_n+(t-1)v_n}{2u_n+(2t-1)v_n}\\ &= t-1+\cfrac{1}{2+\cfrac{1}{t-1+\cfrac{u_n}{v_n}}}\end{aligned}$$

as

$$\begin{aligned}t-1+\frac{u_n}{v_n} &= t-1+t-1+\cfrac{1}{2+\cfrac{1}{\cdots\cdots+\cfrac{1}{2t-2+\cfrac{1}{1+\cfrac{1}{3}}}}}\\ &= 2t-2+\cfrac{1}{2+\cfrac{1}{2t-2+\cfrac{1}{\cdots+\cfrac{1}{2t-2+\cfrac{1}{1+\cfrac{1}{3}}}}}}\end{aligned}$$

we get

$$\begin{aligned}\frac{u_{n+1}}{v_{n+1}} &= t-1+\cfrac{1}{2+\cfrac{1}{2t-2+\cfrac{1}{2+\cfrac{1}{2t-2+\cfrac{1}{\cdots+\cfrac{1}{2t-2+\cfrac{1}{1+\cfrac{1}{3}}}}}}}}\\ &= \left[t-1;\underbrace{2,2t-2,\cdots,2,2t-2}_{n\ times},1,3\right].\end{aligned}$$

completing the proof.

As we reported above, the Diophantine equation E could be transformed into the Diophantine equation $\widetilde{E}$ via the transformation T. Also, we showed that $x = u + 8t - 2$ and $y = v + 8$. So, we can retransfer all results from $\widetilde{E}$ to E by applying the inverse of T. Thus, we can give the following main theorem

Theorem 3.2 *Let D be the Diophantine equation in (1). Then*
(1) The fundamental (minimal) solution of E is $(x_1, y_1) = (16t - 4, 16)$
(2) Define the sequence $\{(x_n, y_n)\}_{n\geq 1} = \{(u_n+8t-2, v_n+8)\}$, where $\{(x_n, y_n)\}$ defined in (5). Then (x_n, y_n) is a solution of E. So it has infinitely many integer solutions $(x_n, y_n) \in \mathbb{Z} \times \mathbb{Z}$.
(3) The solutions (x_n, y_n) satisfy the recurrence relations

$$\begin{cases} x_n = (2t-1)x_{n-1} + (2t^2 - 2t)y_{n-1} - 32t^2 + 36t - 4 \\ \\ y_n = 2x_{n-1} + (2t-1)y_{n-1} - 32t + 20 \end{cases} \tag{11}$$

for $n \geq 2$.

(4) The solutions (u_n, v_n) satisfy the recurrence relations

$$\begin{cases} x_n = (4t-3)(x_{n-1} + x_{n-2}) - x_{n-3} - 64t^2 + 80t - 16 \\ \\ y_n = (4t-3)(y_{n-1} + y_{n-2}) - y_{n-3} - 64t + 64. \end{cases} \tag{12}$$

for $n \geq 4$.

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References

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