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A Generalization of Some Inequalities for the log-Convex Functions

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Abstract

In this paper, we obtain a generalization of a double inequality on the log-convex functions, obtained by E. Neuman [8]. Also, we rediscover some inequalities of gamma and $q-$gamma functions. Finally, we present some new inequalities of Riemann zeta function.

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1 Introduction

The collection of log convex functions features pretty interesting functions that are widely used in applied mathematics and in statistics. [4], [9]. They are often useful in finding bounds for special functions and their zeros. Many inequalities for special functions are statements about the monotonicity of certain quantities [3], [11], [12]. In this work, we will present a monotonicity property and some inequalities of the logarithmically convex functions. As special cases, we will get some inequalities involving the gamma function and the $q-$gamma function.

Recall [2] that if $f(x)$ is a twice differentiable function then $f(x)$ is convex iff $f''(x) \geq 0$ for all x of our interval and hence $f'(x)$ is increasing function in our interval. Also, a function $f(x)$ defined and positive on a certain interval is called log-convex if the function $\log f(x)$ is convex. The condition that $f(x)$ be positive is obviously necessary, for otherwise the function $\log f(x)$ could not be formed.

2 Main result.

Theorem 2.1 *Let $f(x)$ be a positive, differentiable and log-convex function defined on $[0, \infty)$. Let*

$$g(x) = \frac{[f(cx+r)]^a}{[f(bx+s)]^k}; \qquad a, c, r, k, b, s > 0 \tag{1}$$

Then
(1) $g(x)$ decreases on its domain if $c \leq b$, $r \leq s$ and $ac \leq kb$. Hence

$$\frac{[f(cy+r)]^a}{[f(by+s)]^k} \leq \frac{[f(cx+r)]^a}{[f(bx+s)]^k} \leq \frac{[f(r)]^a}{[f(s)]^k} \qquad \forall\ 0 \leq x \leq y. \tag{2}$$

(2) $g(x)$ increases on its domain if $c \geq b$, $r \geq s$ and $ac \geq kb$. Hence

$$\frac{[f(r)]^a}{[f(s)]^k} \leq \frac{[f(cx+r)]^a}{[f(bx+s)]^k} \leq \frac{[f(cy+r)]^a}{[f(by+s)]^k} \qquad \forall\ 0 \leq x \leq y. \tag{3}$$

Proof. *Logarithmic convexity of the function f implies that its logarithmic derivative $\alpha(x) = \frac{f'(x)}{f(x)}$ is increasing function. Then*

$$\alpha(x) \leq \alpha(y) \qquad \forall\ 0 \leq x \leq y. \tag{4}$$

Also,

$$\frac{d}{dx} g(x) = g(x)[ac\,\alpha(cx+r) - kb\,\alpha(bx+s)]. \tag{5}$$

In case of $c \leq b$, $r \leq s$ and $ac \leq kb$, we get for $x \geq 0$ that

$$ac\,\alpha(cx+r) \leq kb\,\alpha(bx+s). \tag{6}$$

Then $g'(x) \leq 0$ and hence $g(x)$ is decreasing on its domain. In particular

$$g(y) \leq g(x) \leq g(0) \qquad \forall\ 0 \leq x \leq y, \tag{7}$$

which is the inequality (2). The second part has similar proof.

3 Some particular cases.

3.1 Inequalities involving the gamma function

The Euler gamma function $\Gamma(x)$ is log-convex function for $x > 0$ and is defined by [2]

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad x > 0,$$

By using a geometrical method, C. Alsina and M. S. Tomás [1] studied an inequality involving the gamma function and proved the following double inequality

$$\frac{1}{n!} \leq \frac{[\Gamma(x+1)]^n}{\Gamma(1+nx)} \leq 1; \qquad 0 \leq x \leq 1,\ n \in N. \tag{8}$$

J. Sándor [10] extended this result to a more general case, and obtained the following inequality

$$\frac{1}{\Gamma(1+a)} \leq \frac{[\Gamma(x+1)]^a}{\Gamma(1+ax)} \leq 1; \qquad 0 \leq x \leq 1,\ a \geq 1 \tag{9}$$

by using the series representation of the digamma function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. L. Bougoffa [3] used the same method of Sándor [10] to prove that the function

$$f_1(x) = \frac{[\Gamma(cx+1)]^a}{[\Gamma(bx+1)]^k}; \qquad x \geq 0,\ a \geq c > 0, \tag{10}$$

is decreasing, so the following double inequality satisfies

$$\frac{[\Gamma(c+1)]^a}{[\Gamma(b+1)]^k} \leq \frac{[\Gamma(cx+1)]^a}{[\Gamma(bx+1)]^k} \leq 1; \qquad 0 \leq x \leq 1,\ a \geq c > 0. \tag{11}$$

Also, A. S. Shabani [11], used the same method of Sándor [10] to prove that the function

$$f_2(x) = \frac{[\Gamma(cx+r)]^a}{[\Gamma(bx+s)]^k}; \qquad x \geq 0, \tag{12}$$

where a, b, c, r, s, k are real numbers such that $0 < cx+r \leq bx+s$, $bk \geq ca > 0$ and $\psi(cx+r) > 0$ (or $\psi(bx+s) > 0$) is decreasing and for $0 \leq x \leq 1$ the following double inequality holds

$$\frac{[\Gamma(c+r)]^a}{[\Gamma(b+s)]^k} \leq \frac{[\Gamma(cx+r)]^a}{[\Gamma(bx+s)]^k} \leq \frac{[\Gamma(r)]^a}{[\Gamma(s)]^k}. \tag{13}$$

Also, in case $ca \geq bk > 0$ and $\psi(cx+r) < 0$ (or $\psi(bx+s) < 0$) the function $f_2(x)$ is decreasing and the inequality (13) holds.

Remark 1: Considering (2) with $f(x) = \Gamma(x)$, $a = b = n \in N$, $y = c = r = s = k = 1$ we obtain inequality (8).
Remark 2: Considering (2) with $f(x) = \Gamma(x)$, $b = a$, $y = c = r = s = k = 1$ we obtain inequality (9).
Remark 3: Considering (2) with $f(x) = \Gamma(x)$, $y = r = s = 1$ we obtain the inequality (11) and the function $f_1(x)$ will be decreasing .
Remark 4: Considering (2) with $f(x) = \Gamma(x)$, $y = 1$ we obtain the inequality (13) and the function $f_2(x)$ will be decreasing, where the conditions $0 < cx + r \leq bx + s$ and $bk \geq ca > 0$ will satisfy $\forall\ x \geq 0$ iff $a, c, r, k, b, s > 0$ also, the condition $\psi(cx + r) > 0$ (or $\psi(bx + s) > 0$) will satisfies automatically because if $f(x) = \Gamma(x)$ then $\psi(x) = \alpha(x)$ which is increasing function.

Lemma 3.1
1− *For all* $0 \leq x \leq y$, $c \leq b$, $r \leq s$, $ac \leq kb$ *and* $a, c, r, k, b, s > 0$, *we have*

$$\frac{[\Gamma(cy+r)]^a}{[\Gamma(by+s)]^k} \leq \frac{[\Gamma(cx+r)]^a}{[\Gamma(bx+s)]^k} \leq \frac{[\Gamma(r)]^a}{[\Gamma(s)]^k}. \tag{14}$$

2− *For all* $0 \leq x \geq y$, $c \geq b$, $r \geq s$, $ac \geq kb$ *and* $a, c, r, k, b, s > 0$, *we have*

$$\frac{[\Gamma(r)]^a}{[\Gamma(s)]^k} \leq \frac{[\Gamma(cx+r)]^a}{[\Gamma(bx+s)]^k} \leq \frac{[\Gamma(cy+r)]^a}{[\Gamma(by+s)]^k}. \tag{15}$$

3.2 Inequalities involving the $q-$gamma function

The $q-$gamma function $\Gamma_q(x)$ is log-convex function for $x > 0$ and is defined by [5]

$$\Gamma_q(x) = \frac{(q,q)_\infty}{(q^x,q)_\infty}(1-q)^{1-x}, \qquad 0 < q < 1,$$

where the $q-$shifted factorials $(a;q)_\infty = \prod_{i=0}^{\infty}(1 - aq^i)$. This function is a $q-$analogue of the gamma function since we have $\lim_{q\to 1} \Gamma_q(x) = \Gamma(x)$.

T. Kim and C. Adiga [6] proved for $0 < q < 1$

$$\frac{1}{\Gamma_q(1+a)} \leq \frac{[\Gamma_q(x+1)]^a}{\Gamma_q(1+ax)} \leq 1; \qquad 0 \leq x \leq 1,\ a \geq 1, \tag{16}$$

Also, T. Mansour [7] proved for $0 < q < 1$

$$\frac{[\Gamma_q(r)]^a}{[\Gamma_q(s)]^k} \leq \frac{[\Gamma_q(sx+r)]^a}{[\Gamma_q(rx+s)]^k} \leq [\Gamma_q(s+r)]^{a-k}, \tag{17}$$

where $\psi_q(bx + s) > 0$, $sa \geq rk$, $0 \leq x \leq 1$, $r \geq s > 0$ and c, d are positive real numbers.
Also, A. S. Shabani [12], proved that the function

$$f_3(x) = \frac{[\Gamma_q(cx+r)]^a}{[\Gamma_q(bx+s)]^k}; \qquad x \geq 0, \tag{18}$$

where a, b, c, r, s, k are real numbers such that $0 < cx+r \leq bx+s$, $bk \geq ca > 0$ and $\psi_q(cx+r) > 0$ (or $\psi_q(bx+s) > 0$) is decreasing and for $0 \leq x \leq 1$ the following double inequality holds

$$\frac{[\Gamma_q(c+r)]^a}{[\Gamma_q(b+s)]^k} \leq \frac{[\Gamma_q(cx+r)]^a}{[\Gamma_q(bx+s)]^k} \leq \frac{[\Gamma_q(r)]^a}{[\Gamma_q(s)]^k}. \tag{19}$$

Also, in case $ca \geq bk > 0$ and $\psi_q(cx+r) < 0$ (or $\psi_q(bx+s) < 0$) the function $f_3(x)$ is decreasing and inequality (19) holds.

Remark 5: Considering (2) with $f(x) = \Gamma_q(x)$, $b = a$, $y = c = r = s = k = 1$ we obtain inequality (16).
Remark 6: Considering (3) with $f(x) = \Gamma_q(x)$, $b = r$, $c = s$, $y = 1$ we obtain inequality (17).
Remark 7: Considering (2) with $f(x) = \Gamma_q(x)$, $y = 1$ we obtain inequality (19) and the function $f_3(x)$ will be decreasing, where the conditions $0 < cx + r \leq bx + s$ and $bk \geq ca > 0$ will satisfy $\forall\ x \geq 0$ iff $a, c, r, k, b, s > 0$ also, the condition $\psi(cx+r) > 0$ (or $\psi(bx+s) > 0$) will be satisfied automatically because if $f(x) = \Gamma_q(x)$ then $\psi_q(x) = \alpha(x)$ which is an increasing function.

Lemma 3.2
1− *For all* $0 \leq x \leq y$, $c \leq b$, $r \leq s$, $ac \leq kb$ *and* $a, c, r, k, b, s > 0$, *we have*

$$\frac{[\Gamma_q(cy+r)]^a}{[\Gamma_q(by+s)]^k} \leq \frac{[\Gamma_q(cx+r)]^a}{[\Gamma_q(bx+s)]^k} \leq \frac{[\Gamma_q(r)]^a}{[\Gamma_q(s)]^k}. \tag{20}$$

2− *For all* $0 \leq x \geq y$, $c \geq b$, $r \geq s$, $ac \geq kb$ *and* $a, c, r, k, b, s > 0$, *we have*

$$\frac{[\Gamma_q(r)]^a}{[\Gamma_q(s)]^k} \leq \frac{[\Gamma_q(cx+r)]^a}{[\Gamma_q(bx+s)]^k} \leq \frac{[\Gamma_q(cy+r)]^a}{[\Gamma_q(by+s)]^k}. \tag{21}$$

3.3 Inequalities involving the Riemann zeta function

The gamma function and the Riemann function are connected with relation [2]

$$\zeta(x+1)\Gamma(x+1) = \int_0^\infty \frac{t^x}{e^x - 1}\, dx \qquad x > 0,$$

where the Riemann zeta function is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \qquad x > 1.$$

The function $\zeta(x+1)\Gamma(x+1)$ is a positive, differentiable and log-convex function for all $x > 0$ [8]. Then, we get the following Lemma

Lemma 3.3
1− *For all* $0 < x \le y$, $c \le b$, $r \le s$, $ac \le kb$ *and* $a, c, r, k, b, s > 0$, *we have*

$$\frac{[\Gamma(cy+r+1)\zeta(cy+r+1)]^a}{[\Gamma(by+s+1)\zeta(by+s+1)]^k} \le \frac{[\Gamma(cx+r+1)\zeta(cx+r+1)]^a}{[\Gamma(bx+s+1)\zeta(bx+s+1)]^k} \le \frac{[\Gamma(r+1)\zeta(r+1)]^a}{[\Gamma(s+1)\zeta(s+1)]^k} \tag{22}$$

2− *For all* $0 < x \le y$, $c \ge b$, $r \ge s$, $ac \ge kb$ *and* $a, c, r, k, b, s > 0$, *we have*

$$\frac{[\Gamma(r+1)\zeta(r+1)]^a}{[\Gamma(s+1)\zeta(s+1)]^k} \le \frac{[\Gamma(cx+r+1)\zeta(cx+r+1)]^a}{[\Gamma(bx+s+1)\zeta(bx+s+1)]^k} \le \frac{[\Gamma(cy+r+1)\zeta(cy+r+1)]^a}{[\Gamma(by+s+1)\zeta(by+s+1)]^k}. \tag{23}$$

3.3.1 Some special cases

By using (14), we obtain

$$\frac{[\Gamma(cx+r+1)\zeta(cx+r+1)]^a}{[\Gamma(bx+s+1)\zeta(bx+s+1)]^k} \le \frac{[\Gamma(r+1)\zeta(cx+r+1)]^a}{[\Gamma(s+1)\zeta(bx+s+1)]^k}, \tag{24}$$

for all $x > 0$, $0 < c \le b$, $0 \le r \le s$, $0 < ac \le kb$.
Also, (22) and (24) give us that

$$\frac{[\Gamma(cy+r+1)\zeta(cy+r+1)]^a}{[\Gamma(by+s+1)\zeta(by+s+1)]^k} \le \frac{[\Gamma(r+1)\zeta(cx+r+1)]^a}{[\Gamma(s+1)\zeta(bx+s+1)]^k}, \tag{25}$$

for all $0 < x \le y$, $0 < c \le b$, $0 \le r \le s$, $0 < ac \le kb$.
If we put $y = c = 1$ and $r = s = 0$ in (25), then we get

$$\frac{\left[\frac{\pi^2}{6}\right]^a}{[\Gamma(b+1)]^k} \le \frac{[\zeta(x+1)]^a}{[\zeta(bx+1)]^k}[\zeta(b+1)]^k, \quad \forall\, 0 < x \le 1;\ 1 \le b;\ 0 < a \le kb. \tag{26}$$

where $\Gamma(2) = 1$ and $\zeta(2) = \frac{\pi^2}{6}$. Now, put $a = k = 1$ in (26), we obtain

$$\zeta(bx+1) \le \frac{\zeta(x+1)\zeta(b+1)\Gamma(b+1)}{\pi^2/6}, \quad \forall\, 0 < x \le 1;\ 1 \le b. \tag{27}$$

Also, put $b = 1$ in (26), to get

$$\left[\frac{\pi^2}{6}\right]^{a-k} \le [\zeta(x+1)]^{a-k}, \quad \forall\, 0 < x \le 1;\ 0 < a \le k, \tag{28}$$

Hence

$$\zeta(x+1) \le \frac{\pi^2}{6}, \qquad \forall\, 0 < x \le 1 \tag{29}$$

with equality if $x = 1$.

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