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Divisibility of Sums of Powers of Odd Integers

T. Aaron Gulliver

Department of Electrical and Computer Engineering
University of Victoria, P.O. Box 3055 STN CSC
Victoria, BC V8W 3P6, Canada
agullive@ece.uvic.ca

Abstract

A number of sequences based on sums of powers of integers is presented. This approach provides a simple derivation of some well known sequences, as well as the construction of many new sequences.

The sums of powers of integers have been the subject of much research over the years. This has led to many well-known combinatorial expressions such as

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}, \tag{1}$$

for the sum of the first n natural numbers and

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\sum_{i=1}^{n} i\right)^2. \tag{2}$$

In this paper we investigate the sums themselves, and in particular the integer sequences that result. Consider the sum of the first n m-th powers

$$\sum_{i=1}^{n} i^m \tag{3}$$

For $n = 4$, the values are

$$\begin{gathered} 10, 30, 100, 354, 1300, 4890, 18700, 72354, 282340, \\ 1108650, 4373500, 17312754, \ldots \end{gathered} \tag{4}$$

This is sequence A103438 in the online encyclopedia of integer sequences maintained by Sloane [1]. Sequences with other values of n and m were considered in [2]. In this paper, we generalize (3) to odd values of i, giving

$$\sum_{i=1}^{n}(2i-1)^m \tag{5}$$

For $n = 1$, we have $1, 1, 1, 1, \ldots$ and for $n = 2$, the sequence is $4, 10, 28, 82, 244, 730, \ldots$ (sequence A034472 in [1]). It can easily be shown that the mth element in the latter sequence is given by $3^m + 1$. Looking at these numbers, one will note that the units digit repeats in a sequence 4, 0, 8, 2. For $n = 3$ to 10, the values of (5) are

$$\begin{array}{ccccccc} 9, & 35, & 153, & 707, & 3369, & 16355, & \ldots \\ 16, & 84, & 496, & 3108, & 20176, & 134004, & \ldots \\ 25, & 165, & 1225, & 9669, & 79225, & 665445, & \ldots \\ 36, & 286, & 2556, & 24310, & 240276, & 2437006, & \ldots \\ 49, & 455, & 4753, & 52871, & 611569, & 7263815, & \ldots \\ 64, & 680, & 8128, & 103496, & 1370944, & 18654440, & \ldots \\ 81, & 969, & 13041, & 187017, & 2790801, & 42792009, & \ldots \\ 100, & 1330, & 19900, & 317338, & 5266900, & 89837890, & \ldots \end{array} \tag{6}$$

The first four of these sequences are A074507, A134006, A134007 and A134008, respectively [1]. The other sequences are new. The third row of (6), shows a regularity in the units digit, namely

$$5 \mid \sum_{i=1}^{5}(2i-1)^m \iff m \neq 0 \bmod 4. \tag{7}$$

and the last row shows that

$$10 \mid \sum_{i=1}^{10}(2i-1)^m \iff m \neq 0 \bmod 4. \tag{8}$$

Note that as with $n = 2$, the units digit repeats every fourth element. In order to show this more clearly, the values of (5) for $n = 1$ to 10 (number of terms) and $m = 1$ to 12 (power) reduced modulo 10 are given in Table 1.

To prove the results above, one could consider expressions such as

$$\sum_{i=1}^{n} 2i - 1 = 1 + 3 + 5 + \ldots + 2n - 1 = n^2, \tag{9}$$

and

$$\sum_{i=1}^{n}(2i-1)^2 \;=\; \frac{1}{3}n(2n-1)(2n+1), \tag{10}$$

Table 1:

power m	number of terms n 1	2	3	4	5	6	7	8	9	10
1	1	4	9	6	5	6	9	4	1	0
2	1	0	5	4	5	6	5	0	9	0
3	1	8	3	6	5	6	3	8	1	0
4	1	2	7	8	9	0	1	6	7	8
5	1	4	9	6	5	6	9	4	1	0
6	1	0	5	4	5	6	5	0	9	0
7	1	8	3	6	5	6	3	8	1	0
8	1	2	7	8	9	0	1	6	7	8
9	1	4	9	6	5	6	9	4	1	0
10	1	0	5	4	5	6	5	0	9	0
11	1	8	3	6	5	6	3	8	1	0
12	1	2	7	8	9	0	1	6	7	8

$$\sum_{i=1}^{n}(2i-1)^3 = n^2(2n^2-1), \tag{11}$$

$$\sum_{i=1}^{n}(2i-1)^4 = \frac{1}{15}n(2n-1)(2n+1)(12n^2-7), \tag{12}$$

etc. Fortunately, there is a much simpler way.

Consider the residues modulo 10 of the powers of the integers $1, 3, 5, 7$ and 9, which are given in Table 2.

Table 2:

power	integer 1	3	5	7	9
1	1	3	5	7	9
2	1	9	5	9	1
3	1	7	5	3	9
4	1	1	5	1	1
5	1	3	5	7	9
6	1	9	5	9	1
⋮	⋮	⋮	⋮	⋮	⋮

The periods of these residues are:

1,5	period 1
9	period 2
3,7	period 4

which are all factors of $\phi(10) = 4$. These values show that the periods of the units digits of (5) must be 4. Determining the units digits for the sums in the table can simply be done by summing the first n columns of Table 2 (taking columns modulo 5), which are given in Table 3. Since Table 3 will repeat for powers greater than 4, this proves the expressions above.

Table 3:

power	number of terms n									
m	1	2	3	4	5	6	7	8	9	$\cdots$
1	1	4	9	6	5	6	9	4	1	$\cdots$
2	1	0	5	4	5	6	5	0	9	$\cdots$
3	1	8	3	6	5	6	3	8	1	$\cdots$
4	1	2	7	8	9	0	1	6	7	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

For other residues, it is a simple matter to form the tables and determine how the sequences of units digits repeat. For example, taking the values of (5) modulo 3 gives Table 4. Note that in this case columns 1 and 2, 4 and 5, and 7 and 8, are identical. A simple proof of this requires only the residues modulo 3 of the powers of the odd integers, which are given in Table 5. This shows that the periods of the residues is either 1 or 2, which is expected since $\phi(3) = 2$. In addition, the columns repeat with every third column all zero, which accounts for the repeated columns in Table 4. In terms of divisibility, Table 4 shows that for $n = 3$ and 6

$$3 \mid \sum_{i=1}^{n}(2i-1)^m,$$

for m odd, and for $n = 4$ and 5,

$$3 \mid \sum_{i=1}^{n}(2i-1)^m,$$

for m even. In addition, for $n = 9$

$$3 \mid \sum_{i=1}^{n}(2i-1)^m.$$

Table 4:

power	number of terms n									
m	1	2	3	4	5	6	7	8	9	10
1	1	1	0	1	1	0	1	1	0	1
2	1	1	2	0	0	1	2	2	0	1
3	1	1	0	1	1	0	1	1	0	1
4	1	1	2	0	0	1	2	2	0	1
5	1	1	0	1	1	0	1	1	0	1
6	1	1	2	0	0	1	2	2	0	1
7	1	1	0	1	1	0	1	1	0	1
8	1	1	2	0	0	1	2	2	0	1
9	1	1	0	1	1	0	1	1	0	1
10	1	1	2	0	0	1	2	2	0	1
11	1	1	0	1	1	0	1	1	0	1
12	1	1	2	0	0	1	2	2	0	1

Another interesting case is the residues modulo 5, which are given in Table 6 for up to 12 terms. The all zeros column occurs for $n = 25$, after which the columns repeat, i.e., the columns for $n = 1$ and $n = 26$ are the same. Note that columns 5 and 10 contain mostly zeros. From this table it is obvious that for $n = 0 \bmod 5$

$$5 \mid \sum_{i=1}^{n} (2i-1)^m \iff m \neq 0 \bmod 4.$$

This is confirmed by the results in Table 7, which shows the residues modulo 5 of the powers of the odd integers, $(2i-1)^m$.

Table 5:

	integer								
power	1	3	5	7	9	11	13	15	17
1	1	0	2	1	0	2	1	0	2
2	1	0	1	1	0	1	1	0	1
3	1	0	2	1	0	2	1	0	2
4	1	0	1	1	0	1	1	0	1
5	1	0	2	1	0	2	1	0	2
6	1	0	1	1	0	1	1	0	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

It is evident that for $i = 1$ or 3 modulo 5, the period is 1, for $i = 0$ modulo 5, the period is 2, and otherwise the period is 4. These values are all factors of $\phi(5) = 4$.

It is left to the reader to determine the results modulo other values, in particular, how does Table 4 differ from the values taken modulo 7? The period of the integer powers will be a factor of $\phi(7) = 6$.

It is much more difficult to determine the divisibility when the values of (5) are considered with m fixed. However, (10) shows that

$$5 \mid \sum_{i=1}^{n}(2i-1)^2,$$

when $n = 5r, 5r-2$ or $5r-3$, $r \geq 1$, and (11) shows that

$$5 \mid \sum_{i=1}^{n}(2i-1)^3,$$

only when $n = 5r$. The proof comes from the fact that 5 is not a factor of the denominator of the constant in (10), and $2n^1 - 1$ in (11) never produces a multiple of 5. The divisibility by other primes is easily established.

More can be said for $m = 2$. Since the constant is $1/3$, and $2n - 1$ and $2n + 1$ are always odd

$$4 \mid \sum_{i=1}^{n}(2i-1)^2,$$

only when $n = 4r$. This is also true for $m = 4$ using similar arguments. In addition, for $m = 3$

$$4 \mid \sum_{i=1}^{n}(2i-1)^3,$$

when $n = 2r$.

Table 6:

power m	number of terms n: 1	2	3	4	5	6	7	8	9	10	11	12
1	1	4	4	1	0	1	4	4	1	0	1	4
2	1	0	0	4	0	1	0	0	4	0	1	0
3	1	3	3	1	0	1	3	3	1	0	1	3
4	1	2	2	3	4	0	1	1	2	3	4	0
5	1	4	4	1	0	1	4	4	1	0	1	4
6	1	0	0	4	0	1	0	0	4	0	1	0
7	1	3	3	1	0	1	3	3	1	0	1	3
8	1	2	2	3	4	0	1	1	2	3	4	0
9	1	4	4	1	0	1	4	4	1	0	1	4
10	1	0	0	4	0	1	0	0	4	0	1	0
11	1	3	3	1	0	1	3	3	1	0	1	3
12	1	2	2	3	4	0	1	1	2	3	4	0

Many other divisibility identities can be established for fixed n or m simply by looking at the sums modulo a number, or by examining the closed form expressions for

$$\sum_{i=1}^{n}(2i-1)^m,$$

for fixed m. In addition, one could consider

$$\sum_{i=1}^{n}(pi-1)^m,$$

for p other than 2.

Table 7:

power	1	3	5	7	9	11	13	15	17	19
1	1	3	0	2	4	1	3	0	2	4
2	1	4	0	4	1	1	4	0	4	1
3	1	2	0	3	4	1	2	0	3	4
4	1	1	0	1	1	1	1	0	1	1
5	1	3	0	2	4	1	3	0	2	4
6	1	4	0	4	1	1	4	0	4	1
7	1	2	0	3	4	1	2	0	3	4
8	1	1	0	1	1	1	1	0	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

References

[1] N.J.A. Sloane, On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/index.html.

[2] T.A. Gulliver, Divisibility of sums of powers of integers, *Int. Math. J.*, Vol. 3, No. 7, pp. 699-704, Apr. 2003.

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