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Weak Convergence Theorem for an Equilibrium Problem and a Nonexpansive Semigroup in Hilbert Spaces

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Abstract

In this paper, we introduce a weak convergent iterative process for finding a common element of the solution set for an equilibrium problem and the set of fixed points for a nonexpansive semigroup in Hilbert spaces.

Keywords: Common fixed points, nonexpansive mapping, and bifunction

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1. Introduction

Let H be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle ., . \rangle$ and $\|.\|$, respectively, and let C be a nonempty closed and convex subset of H. Denote the metric projection from $x \in H$ onto C by $P_C(x)$. Let T be a nonexpansive mapping on C, i.e., $T : C \to C$ and $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. We know that $F(T)$ is nonempty, if C is bounded, for more details see [1].

For finding an element $p \in F(T)$, where T is a nonexpansive mapping on C, Mann [2] introduced in 1953 the following iteration process:

$$\begin{aligned} x_0 &\in C \quad \text{any element,} \\ x_{k+1} &= \alpha_k x_k + (1-\alpha_k)Tx_k, \end{aligned} \tag{1.1}$$

where $\{\alpha_k\}$ is a sequence of positive real numbers. Processes (1.1) has only weak convergence, in general (see [3] for an example).

Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem which is for a bifunction $G(u, v)$ defined on $C \times C$ to find $u^* \in C$ such that

$$G(u^*, v) \geq 0 \quad \forall v \in C. \tag{1.2}$$

The bifunction G is assumed to satisfy the following set of standard properties.
Condition 1.1:

(A1) $G(u, u) = 0 \quad \forall u \in C$;
(A2) $G(u, v) + G(v, u) \leq 0 \quad \forall (u, v) \in C \times C$;
(A3) For every $u \in C$, $G(u, .) : C \to (-\infty, +\infty)$ is lower semicontinuous and convex;
(A4) $\overline{\lim}_{t\to+0} G((1-t)u + tz, v) \leq G(u, v) \quad \forall (u, z, v) \in C \times C \times C$;

Denote the set of solutions of (1.2) by $EP(G)$. Some methods have been proposed to solve the equilibrium problem; see for instance [4-11].

For finding an element $p \in F(T) \cap EP(G)$, Tada and Takahashi [12] introduced the following weak convergent algorithm:

$$\begin{aligned} & x_0 \in H, \\ & u_k \in C : G(u_k, y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k\rangle \geq 0, \quad \text{for all} \quad y \in C, \\ & x_{k+1} = \alpha_k x_k + (1-\alpha_k)Tu_k, \end{aligned} \tag{1.3}$$

for every $k \geq 0$, where $\{\alpha_k\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_k\} \subset (0, \infty)$ satisfies $\liminf_{k\to\infty} r_k > 0$.

Let $\{T(s) : s > 0$ be a nonexpansive semigroup on C, that is,
(1) for each $s > 0, T(s)$ is a nonexpansive mapping on C;
(2) $T(0)x = x$ for all $x \in C$;
(3) $T(s_1 + s_2) = T(s_1) \circ T(s_2)$ for all $s_1, s_2 > 0$;
(4) for each $x \in C$, the mapping $T(.)x$ from $(0, \infty)$ into C is continuous.

Denote by $\mathcal{F} = \cap_{s\geq 0} F(T(s))$. We know [13] that $\mathcal{F}$ is a closed convex subset in H and $\mathcal{F} \neq \emptyset$ if C is compact (see, [14]).

For finding an element $p \in \mathcal{F}$, Shioji and Takahashi [15] introduced the implicit iteration method

$$x_k = \alpha_k u + (1-\alpha_k)\frac{1}{s_k}\int_0^{s_k} T(s)x_k ds, \quad u \in C, k \geq 0,$$

where $\{\alpha_k\}$ is a sequence in (0,1) and $\{s_k\}$ is a devergent sequence of real positive numbers. Further, Nakajo and Takahashi [13] introduced an iteration procedure for nonexpansive semigroup $\{T(s) : s > 0\}$ as follows:

$$\begin{aligned} x_0 &\in C, \\ y_k &= \alpha_k x_k + (1-\alpha_k)\frac{1}{s_k}\int_0^{s_k} T(s)x_k ds, \\ C_k &= \{z \in C : \|y_k - z\| \leq \|x_k - z\|\}, \\ Q_k &= \{z \in C : \langle x_k - z, x_0 - x_k\rangle \geq 0\}, \\ x_{k+1} &= P_{C_k \cap Q_k}(x_0) \end{aligned} \tag{1.4}$$

for each $k > 0$, where $\alpha_k \in$ [0,a] for some $a \in$ [0,1) and $\{s_k\}$ is a positive real number divergent sequence. Under some conditions on $\{\alpha_k\}$ and $\{s_k\}$, the sequence $\{x_n\}$ defined by (1.4) converges strongly to $P_{\mathcal{F}}(x_0)$.

In 2007, He and Chen [16] considered for the nonexpansive semigroup an iteration procedure:

$$\begin{aligned} x_0 &\in C, \\ y_k &= \alpha_k x_k + (1-\alpha_k)T(s_k)x_k, \\ C_k &= \{z \in C : \|y_k - z\| \leq \|x_k - z\|\}, \\ Q_k &= \{z \in C : \langle x_k - z, x_0 - x_k\rangle \geq 0\}, \\ x_{k+1} &= P_{C_k \cap Q_k}(x_0) \end{aligned} \tag{1.5}$$

for $k \geq 0$, where $\alpha_k \in [0, a)$ for some $a \in [0, 1)$ and $s_k \geq 0, \lim_{k\to\infty} s_k = 0$, then the sequence $\{x_k\}$ in (1.5) converges to $P_{\mathcal{F}}(x_0)$. In 2008 Saejung [17] showed that the proof of the main result in [16] is very questionable and corrected this fact under new condition on s_k: $\liminf_k s_k = 0, \limsup_k s_k > 0$, and $\lim_k(s_{k+1} - s_k) = 0$.

The problem studied in this paper is to find an element

$$p \in \mathcal{F} \cap EP(G), \tag{1.6}$$

assumed to be nonempty, for a nonexpansive semigroup $\{T(s) : s > 0\}$ on C and a bifunction $G(u, v)$ defined on $C \times C$.

To solve (1.6) in the case that $C = H$, very recently, Cianciaruso, Marino and Muglia [18] considered the method

$$\begin{aligned} &x_0 \in H, \\ &x_{n+1} = \alpha_k \gamma f(x_k) + (1 - \alpha_k A) T_k u_k, \\ &u_k \in C: \quad G(u_k, v) + \frac{1}{r_k}\langle u_k - x_k, y - u_k \rangle \geq 0, \quad \forall y \in H, \end{aligned}$$

where

$$T_k x = \frac{1}{s_k}\int_0^{s_k} T(s)x ds, \tag{1.7}$$

f is a α-contraction, and $A : H \to H$ is a strongly positive linear bounded selfadjoint mapping.

Motivated [12], [13] and [16], we obtain weak convergence theorem for the following iteration process:

$$\begin{aligned} &x_0 \in H, \\ &u_k \in C : G(u_k, y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k \rangle \geq 0 \quad \forall y \in C, \\ &x_{k+1} = \mu_k x_k + (1 - \mu_k) T_k u_k, \end{aligned} \tag{1.8}$$

for every $k \geq 0$, where $\{\mu_k\} \subset [a, b]$ for some $a, b \in (0, 1)$, $\{r_k\} \subset (0, \infty)$ satisfies $\liminf_{k\to\infty} r_k > 0$, and T_k is defined by

$$\begin{aligned} &T_k x = T(s_k)x \quad \forall x \in C, \\ &\liminf_{k\to\infty} s_k = 0, \limsup_{k\to\infty} s_k > 0, \lim_{k\to\infty}(s_{k+1} - s_k) = 0 \end{aligned} \tag{1.9}$$

or T_k is defined by (1.7) with

$$\lim_{k\to\infty} s_k = \infty. \tag{1.10}$$

The weak convergence of (1.8)-(1.10) is proved in the next section.

2. Main result

We need the following facts in the proof of our results.

Definition 1.1. A Banach space E is said to satisfy Opial's condition [19] if whenever $\{x_k\}$ is a sequence in E which converges weakly to x, as $k \to \infty$, then

$$\limsup_{k\to\infty} \|x_k - x\| < \limsup_{k\to\infty} \|x_k - y\|, \quad \forall y \in E \quad \text{with } x \neq y.$$

It is well known that Hilbert space and $l^p (1 < p < \infty)$ satisfy Opial's condition [20].

Lemma 2.2 [21] *Let $\{\mu_k\}$ be a sequence of real numbers such that $0 < a \leq \mu_k \leq b < 1$ for all $k \geq 0$. Let $\{v_k\}$ and $\{w_k\}$ be sequences of H such that*

$$\limsup_{k\to\infty} \|v_n\| \leq c, \quad \limsup_{k\to\infty} \|w_n\| \leq c,$$
$$\lim_{k\to\infty} \|\mu_k v_n + (1-\mu_k)w_k\| = c.$$

Then, $\lim_{k\to\infty} \|v_k - w_k\| = 0$.

Lemma 2.3 [21]. *Let C be a nonempty closed convex subset of H. Let $\{x_k\}$ be sequences of H. Suppose that, for all $y \in C$,*

$$\|x_{k+1} - y\| \leq \|x_k - y\|,$$

for every $k \geq 0$. Then, $\{P_C(x_k)\}$ converges strongly to some $p \in C$.

Lemma 2.4 [22]. *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique $z \in C$ such that $\|z - x\| \leq \|y - x\|$ for all $y \in C$, and $z \in P_C(x)$ if and only if $\langle z - x, y - z\rangle \geq 0$ for all $y \in C$, where P_C is the metric projection of H onto C.*

Lemma 2.5 [7]. *Let C be a nonempty closed convex subset of H and G be a bifunction of $C \times C$ into $(-\infty, +\infty)$ satisfying condition 1.1. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$G(z, v) + \frac{1}{r}\langle z - x, v - z\rangle \geq 0 \quad \forall v \in C.$$

Lemma 2.6 [7]. *Assume that $G : C \times C \to (-\infty, +\infty)$ satisfies condition 1.1. For $r > 0$ and $x \in H$, define a mapping $T^r : H \to C$ as follows:*

$$T^r(x) = \{z \in C : G(z, v) + \frac{1}{r}\langle z - x, v - z\rangle \geq 0 \quad \forall v \in C\}. \tag{2.1}$$

Then, the following statements hold:
(i) T^r is single-valued;
(ii) T^r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T^r(x) - T^r(y)\|^2 \leq \langle T^r(x) - T^r(y), x - y\rangle;$$

(iii) $F(T^r) = EP(G)$;
(iv) $EP(G)$ is closed and convex.

Lemma 2.7 [23]. *Let C be a nonempty bounded closed convex subset of H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C. Then, for any $h \geq 0$*

$$\lim_{t\to\infty} \sup_{y\in C} \left\| T(h)\left(\frac{1}{t}\int_0^t T(s)y ds\right) - \frac{1}{t}\int_0^t T(s)y ds \right\| = 0.$$

Lemma 2.8 (Demiclosedness Principle) [24]. *If C is a closed convex subset of H, T is a nonexpansive mapping on C, $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x - Tx = 0$.*

Theorem 2.9. *Let C be a nonempty closed convex subset in a real Hilbert space H. Let G be a bifunction from $C \times C$ to $(-\infty, +\infty)$ satisfying conditions (A1)-(A4) and let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} \cap EP(G) \neq \emptyset$. Let $\{x_k\}$ be a sequence generated by (1.8)-(1.10). Then, it converges weakly to an element $p \in \mathcal{F} \cap EP(G)$, where $p = \lim_{k\to\infty} P_{\mathcal{F}\cap EP(G)}(x_k)$.*

Proof. First, we consider the case that T_k is defined by (1.9). By Lemma 2.5, $\{u_k\}$ and $\{x_k\}$ are well defined. For each $u \in \mathcal{F} \cap EP(G)$, by putting $u_k = T^{r_k}x_k$ in (2.1) and using (ii) in Lemma 2.6, we have that

$$\|u_k - u\| = \|T^{r_k}x_k - T^{r_k}u\| \leq \|x_k - u\|.$$

Therefore,

$$\begin{aligned}
\|x_{k+1} - u\| &= \|\mu_k(x_k - u) + (1-\mu_k)(T_k u_k - u)\| \\
&\leq \mu_k\|x_k - u\| + (1-\mu_k)\|T_k u_k - T_k u\| \\
&\leq \mu_k\|x_k - u\| + (1-\mu_k)\|u_k - u\| \\
&\leq \mu_k\|x_k - u\| + (1-\mu_k)\|x_k - u\| \\
&\leq \|x_k - u\|.
\end{aligned}$$

So, there exists

$$\lim_{k\to\infty} \|x_k - u\| = c, \tag{2.2}$$

and hence $\{x_k\}$ and $\{u_k\}$ are bounded.

On the other hand, from (ii) in Lemma 2.6, we have for every $u \in \mathcal{F} \cap EP(G)$ that

$$\begin{aligned}
\|u_k - u\|^2 &= \|T^{r_k}x_k - T^{r_k}u\|^2 \\
&\leq \langle T^{r_k}x_k - T^{r_k}u, x_k - u\rangle \\
&\leq \langle u_k - u, x_k - u\rangle \\
&\leq \frac{1}{2}[\|u_k - u\|^2 + \|x_k - u\|^2 - \|u_k - x_k\|^2].
\end{aligned}$$

Thus,

$$\|u_k - u\|^2 \leq \|x_k - u\|^2 - \|u_k - x_k\|^2.$$

So, by the convexity of $\|.\|^2$, we have

$$\begin{aligned}
\|x_{k+1} - u\|^2 &= \|\mu_k(x_k - u) + (1-\mu_k)(T_k u_k - u)\|^2 \\
&\leq \mu_k\|x_k - u\|^2 + (1-\mu_k)\|T_k u_k - u\|^2 \\
&\leq \mu_k\|x_k - u\|^2 + (1-\mu_k)\|u_k - u\|^2 \\
&\leq \mu_k\|x_k - u\| + (1-\mu_k)(\|x_k - u\|^2 - \|x_k - u_k\|^2) \\
&\leq \|x_k - u\|^2 - (1-\mu_k)\|x_k - u_k\|^2 \\
&\leq \|x_k - u\|^2 - (1-b)\|x_k - u_k\|^2.
\end{aligned}$$

Therefore,

$$(1-b)\|x_k - u_k\|^2 \le \|x_k - u\|^2 - \|x_{k+1} - u\|^2.$$

From (2.2) we get

$$\lim_{k\to\infty} \|x_k - u_k\| = 0. \tag{2.3}$$

Since $\liminf_{k\to\infty} r_k > 0$, we obtain

$$\lim_{k\to\infty} \left\| \frac{x_k - u_k}{r_k} \right\| = 0. \tag{2.4}$$

As $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_j}\}$ of the sequence $\{x_k\}$ which converges weakly to p. From (2.3), we also have that $\{u_{k_j}\}$ converges weakly to p. Since $\{u_{k_j}\} \subset C$, we have that $p \in C$.

We shall show that $p \in EP(G)$. By $u_k = T_{r_k} x_k$, we have

$$G(u_k, y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k \rangle \ge 0 \quad \forall y \in C.$$

From condition (A2), we get

$$\frac{1}{r_k}\langle u_k - x_k, y - u_k \rangle \ge G(y, u_k) \quad \forall y \in C.$$

Therefore,

$$\langle \frac{u_{k_j} - x_{k_j}}{r_{k_j}}, y - u_{k_j} \rangle \ge G(y, u_{k_j}) \quad \forall y \in C.$$

From condition (A3) and (2.4), it implies that

$$0 \ge G(y, p), \quad \forall y \in C.$$

that is equivalent to $G(p, y) \ge 0 \quad \forall y \in C$ (see [10]). It means that $p \in EP(G)$.

Next we show that $p \in \mathcal{F}$. Since $\|T_k u_k - u\| \le \|u_k - u\| \le \|x_k - u\|$ for every $u \in \mathcal{F} \cap EP(G)$, from (2.2) it follows

$$\limsup_{k\to\infty} \|T_k u_k - u\| \le c.$$

Further, we have

$$\lim_{k\to\infty} \|\mu_k(x_k - u) + (1-\mu_k)(T_k u_k - u)\| = \lim_{k\to\infty} \|x_{k+1} - u\| = c.$$

By Lemma 2.2, we obtain

$$\lim_{k\to\infty} \|T_k u_k - x_k\| = 0. \tag{2.5}$$

By virtue of (2.3), (2.5) and

$$\|T_k u_k - u_k\| \le \|T_k u_k - x_k\| + \|x_k - u_k\|,$$

we get

$$\lim_{k\to\infty} \|T_k u_k - u_k\| = 0. \tag{2.6}$$

Without loss of generality, as in [17], let

$$\lim_{j\to\infty} s_{k_j} = \lim_{j\to\infty} \frac{\|T_{k_j} u_{k_j} - u_{k_j}\|}{s_{k_j}} = 0. \tag{2.7}$$

Now, we prove that $p = T(s)p$ for a fixed $s > 0$. It is easy to see that

$$\begin{aligned}
\|u_{k_j} - T(s)p\| \le & \sum_{l=0}^{[s-s_{k_j}]-1} \|T(ls_{k_j})u_{k_j} - T((l+1)s_{k_j})u_{k_j}\| \\
& + \left\|T\left(\left[\frac{s}{s_{k_j}}\right]\right)u_{k_j} - T\left(\left[\frac{s}{s_{k_j}}\right]\right)p\right\| + \left\|T\left(\left[\frac{s}{s_{k_j}}\right]\right)p - T(s)p\right\| \\
& \le \left[\frac{s}{s_{k_j}}\right]\|u_{k_j} - T(s_{k_j})u_{k_j}\| + \|u_{k_j} - p\| + \left\|T\left(t - \left[\frac{s}{s_{k_j}}\right]s_{k_j}\right)p - p\right\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u_{k_j} - T(s)p\| \le & \frac{s}{s_{k_j}}\|u_{k_j} - T(s_{k_j})u_{k_j}\| \\
& + \|u_{k_j} - p\| + \sup\{\|T(s)p - p\| : 0 \le s \le s_{k_j}\}.
\end{aligned}$$

This fact and (2.7) imply that

$$\limsup_{j\to\infty} \|u_{k_j} - T(s)p\| \le \limsup_{j\to\infty} \|u_{k_j} - p\|.$$

As every Hilbert space satisfies Opial's condition, we have $T(s)p = p$. Therefore, $p \in \mathcal{F}$.

Let $\{x_{k_i}\}$ be another subsequence of $\{x_k\}$ convergent weakly to $p' \in \mathcal{F} \cap EP(G)$. If $p \ne p'$, from the Opial theorem [19], we get

$$\begin{aligned}
\lim_{k\to\infty} \|x_k - p\| &= \limsup_{k_j\to\infty} \|x_{k_j} - p\| < \limsup_{j\to\infty} \|x_{k_j} - p'\| \\
&= \lim_{k\to\infty} \|x_k - p'\| = \limsup_{i\to\infty} \|x_{k_i} - p'\| \\
&< \limsup_{i\to\infty} \|x_{k_i} - p\| = \lim_{k\to\infty} \|x_k - p\|.
\end{aligned}$$

This is a contracdition. So, we have $p = p'$. This implies that $\{x_k\}$ converges weakly to $p \in \mathcal{F} \cap EP(G)$.

Let $z_k = P_{\mathcal{F}\cap EP(G)}(x_k)$. Since $p \in \mathcal{F} \cap EP(G)$, by Lemma 2.4, we have

$$\langle x_k - z_k, z_k - p\rangle \geq 0.$$

Using Lemma 2.2, we have that the sequence $\{z_k\}$ converges strongly to some $p_0 \in \mathcal{F} \cap EP(G)$. Since $\{x_k\}$ converges weakly to p, we have

$$\langle p - p_0, p_0 - p\rangle \geq 0.$$

Therefore, we obtain

$$p = p_0 = \lim_{k\to\infty} P_{\mathcal{F}\cap EP(G)}(x_k).$$

Further, in the case that T_k is defined by (1.7) and (1.10), we have only to prove that $p \in \mathcal{F}$ from (2.6). For this purpose, we shall estimate the value $\|u_n - T(h)u_n\|$ for each $h > 0$ as follows.

$$\begin{aligned}
\|T(h)u_n - u_n\| &\leq \left\|T(h)u_n - T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right)\right\| \\
&\quad + \left\|T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right) - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\
&\quad + \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - u_n\right\| \\
&\leq 2\left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - u_n\right\| \\
&\quad + \left\|T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right) - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\|.
\end{aligned} \tag{2.8}$$

By Lemma 2.7 , we get

$$\lim_{n\to\infty}\left\|T(h)\left(\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right) - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| = 0,$$

for every $h \in (0,\infty)$ and hence, by (2.6) and (2.8), we obtain

$$\lim_{n\to\infty} \|T(h)u_n - u_n\| = 0$$

for each $h > 0$ By Lemma 2.8, this implies $p \in F(T(h))$ for all $h > 0$. As for the case (1.9), we also obtain that the sequence $\{x_n\}$ defined by (1.8) with (1.10) converges weakly to p as $n \to \infty$. This completes the proof.

As direct consequences of Theorem 2.7, we can obtain the following corollaries.

Corollary 2.10. *Let C be a nonempty closed convex subset in a real Hilbert space H. Let G be a bifunction from $C \times C$ to $(-\infty, +\infty)$ satisfying conditions (A1)-(A4) and let T be a nonexpansive mapping on C such that $F(T) \cap EP(G) \neq \emptyset$. Let $\{x_k\}$ be a sequence generated by:*

$$\begin{aligned} &x_0 \in H \quad \text{any element}, \\ &u_k \in C : G(u_k, y) + \frac{1}{r_k}\langle u_k - x_k, y - u_k\rangle \geq 0 \quad \forall y \in C, \\ &x_{k+1} = \mu_k x_k + (1 - \mu_k)Tu_k, k \geq 0, \end{aligned}$$

where $\{\mu_k\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_k\} \subset (0, \infty)$ satisfies $\liminf_{k\to\infty} r_k > 0$. Then, the sequence $\{x_k\}$ converges weakly to an element $p \in F(T) \cap EP(G)$, where $p = \lim_{k\to\infty} P_{F(T)\cap EP(G)}(x_k)$.

Proof. By puting $T(s) = T$ for all $s \geq 0$ in Theorem 2.7, we obtain Corollary 2.8.

Note that Corollary 2.10 is Theorem 4.1 in [12].

Corollary 2.11. *Let C be a nonempty closed convex subset in a real Hilbert space H. Let $\{T(s) : s > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} \neq \emptyset$. Let $\{x_k\}$ be a sequence generated by*

$$\begin{aligned} &x_0 \in H \quad \text{any element}, \\ &u_k = P_C(x_k), \\ &x_{k+1} = \mu_k x_k + (1 - \mu_k)T_k u_k, k \geq 0, \end{aligned}$$

where $\{\mu_k\} \subset [a, b]$ for some $a, b \in (0, 1)$ and T_k is defined by (1.9) or (1.10). Then, the sequence $\{x_k\}$ converges weakly to an element $p \in \mathcal{F}$, where $p = \lim_{k\to\infty} P_{\mathcal{F}}(x_k)$.

Proof. Putting $G(u, v) = 0$ for all $u, v \in C$ and $r_k = 1$ in Theorem 2.9, and noticing that

$$\langle u_k - x_k, y - u_k\rangle \geq 0 \quad \forall y \in C \Leftrightarrow \quad u_k = P_C(x_k),$$

we obtain Corollary 2.11.

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