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Best Extension of Hardy-Hilbert's Integral Inequality

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Abstract. We present a generalization of Hardy-Hilbert's integral inequality and deduce some special cases as an application.

$$\int_0^\infty \cdots \int_0^\infty \frac{F_1(u_1)\, f_1'(u_1) \ldots F_n(u_n)\, f_n'(u_n)}{\left(f_1(u_1)+\ldots+f_n(u_n)\right)^\lambda}\, du_1 \ldots du_n$$

$$\leq \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \left(\Gamma\left(2+p_i\left(\lambda_i-1\right)\right) \int_0^\infty \frac{f_i'(u) F_i(u)}{\left(f_i(u)\right)^{2+P_I(\lambda_I-1)}}\, du \right)^{1/p_i} .$$

Other cases are also deduced as an application .

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1. Introduction

If $f, g \geq 0$ are such that

$$0 < \int_0^\infty f^2(x)\, dx < \infty, \quad 0 < \int_0^\infty g^2(x)\, dx < \infty ,$$

then the famous Hilbert's integral inequality is given by

(1.1) $$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x+y}\, dx\, dy < \pi \left(\int_0^\infty f^2(x)\, dx \int_0^\infty g^2(x)\, dx \right)^{1/2} .$$

where the constant factor π is the best possible (see [2]) . Inequality (1) has been generalized by Hardy-Riesz [1] as

If p>1, $\frac{1}{p}+\frac{1}{q}=1$,

$$0<\int_0^\infty f^p(x)\,dx<\infty\ ,\quad 0<\int_0^\infty g^q(x)\,dx<\infty,$$

then

(1.2) $$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy<\frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty g^q(x)\,dx\right)^{1/q},$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. The inequality (2) is called Hardy-Hilbert’s integral inequality, and is important in analysis and its applications (see [3]). Yang [4,5] has extended inequality (2) by proving the following

If $f,g\geq 0,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda>2-\min\{p,q\}$ are such that

$$0<\int_0^\infty x^{1-\lambda}f^p(x)\,dx<\infty,\quad 0<\int_0^\infty x^{1-\lambda}g^q(x)\,dx<\infty\ ,$$

then the extended Hardy-Hilbert's inequality is given by

(1.3) $$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy<k_\lambda(p)\left(\int_0^\infty t^{1-\lambda}f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty t^{1-\lambda}g^q(t)\,dt\right)^{1/q},$$

and the constant $k_\lambda(p)=B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$, where B denotes the beta function, is the best possible. The aim of this paper is to give new kinds of Hardy-Hilbert's integral inequality.

2. Results

We state and prove the following

Theorem 2.1.

Let $f_i\ ,\ f_i',F_i\geq 0,\ f_i(0)=0,\ f_i(\infty)=\infty\ ,\ p_i>1,\ \ i=1,\dots,n\ \ ,\ \sum_{i=1}^n\frac{1}{p_i}=1,\ \lambda>0,$

$\lambda-1=\sum_{i=1}^n\lambda_i\ ,\ p_i(1-\lambda_i)<2$. *Then, we have*

(2.1) $$\int_0^\infty\cdots\int_0^\infty \frac{F_1(u_1)\ f_1'(u_1)\dots F_n(u_n)\ f_n'(u_n)}{\left(f_1(u_1)+\dots+f_n(u_n)\right)^\lambda}\,du_1\dots du_n$$

$$\leq \frac{1}{\Gamma(\lambda)}\prod_{i=1}^{n}\left(\Gamma(2+p_i(\lambda_i-1))\int_0^\infty \frac{f_i'(u)\,F_i(u)}{(f_i(u))^{2+p_i(\lambda_i-1)}}\,du\right)^{1/p_i}.$$

Proof. Define

$$G_i(t) = \int_0^\infty e^{-t\,f_i(u_i)}\, f_i'(u_i)\,F_i(u_i)\,du_i\ .$$

Then, we have

$$I = \int_0^\infty t^{\lambda-1} G_1(t)\ldots G_n(t)\ dt$$

$$= \int_0^\infty t^{\lambda-1}\int_0^\infty e^{-t\,f_1(u_1)}\, f_1'(u_1)\,F_1(u_1)\,du_1\ldots\int_0^\infty e^{-t\,f_n(u_n)}\, f_n'(u_n)\,F_n(u_n)\,du_n dt$$

$$= \int_0^\infty\ldots\int_0^\infty F_1(u_1)\ f_1'(u_1)\ldots F_n(u_n)\ f_n'(u_n)\,du_1\ldots du_n\int_0^\infty t^{\lambda-1}e^{-t\left(f_1(u_1)+\ldots+f_n(u_n)\right)}\ dt$$

Let $t\left(f_1(u_1)+\ldots+f_n(u_n)\right)=v$, gives

$$I = \int_0^\infty\ldots\int_0^\infty F_1(u_1)\ f_1'(u_1)\ldots F_n(u_n)\ f_n'(u_n)\,du_1\ldots du_n\int_0^\infty\left(\frac{v}{f_1(u_1)+\ldots+f_n(u_n)}\right)^{\lambda-1}\times$$

$$e^{-v}\,\frac{dv}{f_1(u_1)+\ldots+f_n(u_n)}$$

$$= \int_0^\infty\ldots\int_0^\infty \frac{F_1(u_1)\ f_1'(u_1)\ldots F_n(u_n)\ f_n'(u_n)}{\left(f_1(u_1)+\ldots+f_n(u_n)\right)^{\lambda}}\,du_1\ldots du_n\int_0^\infty v^{\lambda-1}e^{-v}dv$$

(2.2) $$= \Gamma(\lambda)\int_0^\infty\ldots\int_0^\infty \frac{F_1(u_1)\ f_1'(u_1)\ldots F_n(u_n)\ f_n'(u_n)}{\left(f_1(u_1)+\ldots+f_n(u_n)\right)^{\lambda}}\,du_1\ldots du_n\ .$$

On the other hand

$$I = \int_0^\infty t^{\lambda_1}G_1(t)\ldots t^{\lambda_n}\ G_n(t)\ dt$$

$$\leq \left(\int_0^\infty t^{\lambda_1 p_1}G_1^{p_1}(t)\,dt\right)^{1/p_1}\ldots\left(\int_0^\infty t^{\lambda_n p_n}G_n^{p_n}(t)\,dt\right)^{1/p_n}$$

$$= \prod_{i=1}^{n}\left(\int_0^\infty t^{\lambda_i p_i}G_i^{p_i}(t)\,dt\right)^{1/p_i}.$$

Now,

$$\int_0^\infty t^{\lambda_i p_i} G_i^{p_i}(t)\,dt = \int_0^\infty t^{\lambda_i p_i}\left(\int_0^\infty e^{-t\,f_i(u_i)}\, f_i'(u_i)\,F_i(u_i)\,du_i\right)^{p_i} dt$$

$$\le \int_0^\infty t^{\lambda_i p_i}\left(\int_0^\infty e^{-t\,f_i(u_i)}\, f_i'(u_i)\,F_i^{p_i}(u_i)\,du_i\right)\left(\int_0^\infty e^{-t\,f_i(u_i)}\, f_i'(u_i)\,du_i\right)^{p_i-1} dt$$

$$= \int_0^\infty f_i'(u_i)\,F_i^{p_i}(u_i)\,du_i \int_0^\infty t^{\lambda_i p_i}\left(\left[\frac{1}{t}e^{-t\,f_i(u_i)}\right]_\infty^0\right)^{p_i-1} e^{-t\,f_i(u_i)}\,dt$$

$$= \int_0^\infty f_i'(u_i)\,F_i^{p_i}(u_i)\,du_i \int_0^\infty t^{1+\lambda_i p_i-p_i} e^{-t\,f_i(u_i)}\,dt$$

$$= \int_0^\infty \frac{f_i'(u_i)\,F_i^{p_i}(u_i)}{\left(f_i(u_i)\right)^{2+p_i(\lambda_i-1)}}\,du_i \int_0^\infty \left(t f_i(u_i)\right)^{1+p_i(\lambda_i-1)} e^{-t\,f_i(u_i)}\, f_i(u_i)\,dt$$

$$= \int_0^\infty \frac{f_i'(u_i)\,F_i^{p_i}(u_i)}{\left(f_i(u_i)\right)^{2+p_i(\lambda_i-1)}}\,du_i \int_0^\infty z^{1+p_i(\lambda_i-1)} e^{-z}\,dz$$

$$= \Gamma\left(2+p_i(\lambda_i-1)\right)\int_0^\infty \frac{f_i'(u_i)\,F_i^{p_i}(u_i)}{\left(f_i(u_i)\right)^{2+p_i(\lambda_i-1)}}\,du_i$$

Therefore

$$I \le \prod_{i=1}^n\left(\Gamma\left(2+p_i(\lambda_i-1)\right)\int_0^\infty \frac{f_i'(u_i)\,F_i^{p_i}(u_i)}{\left(f_i(u_i)\right)^{2+p_i(\lambda_i-1)}}\,du_i\right)^{1/p_i}. \tag{2.3}$$

Combining (2.2) and (2.3) we obtain

$$\int_0^\infty\!\!\cdots\!\int_0^\infty \frac{F_1(u_1)\, f_1'(u_1)\ldots F_n(u_n)\, f_n'(u_n)}{\left(f_1(u_1)+\ldots+f_n(u_n)\right)^\lambda}\,du_1\ldots du_n$$

$$\le \frac{1}{\Gamma(\lambda)}\prod_{i=1}^n\left(\Gamma\left(2+p_i(\lambda_i-1)\right)\int_0^\infty \frac{f_i'(u)\,F_i^{p_i}(u)}{\left(f_i(u)\right)^{2+p_i(\lambda_i-1)}}\,du\right)^{1/p_i}.$$

This completes the proof.

Corollary 2.2.

Let $g_i \ge 0,\quad q_i \ge 1,\ i = 1,\ldots,n$. *Let* p_i, λ, λ_i *be as defined in Theorem* 2.1.

Define $G_i(t) = \int_0^t g^{q_i}(t)\,dt$, $G_i(\infty) = \infty$. *Then*

(2.4) $$\int_0^\infty \ldots \int_0^\infty \frac{g_1^{q_1}(t_1)\ldots g_n^{q_n}(t_n)}{\left(G_1(t_1)+\ldots+G_n(t_n)\right)^\lambda} dt_1 \ldots dt_n$$
$$\leq \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \left(\Gamma(2+p_i(\lambda_i-1)) \int_0^\infty \frac{g_i^{q_i}(t_i)}{(G_i(t_i))^{2+p_i(\lambda_i-1)}} \right)^{1/p_i}.$$

Proof. Follows from Theorem 2.1, by putting
$$f_i = G_i, \quad f_i' = g_i^{q_i}, \quad and \quad F_i = 1, \quad i = 1, \ldots, n.$$

Corollary 2.3. *Let the assumption of Theorem* 2.1 *be satisfied. Then*

(2.5) $$\int_0^\infty \ldots \int_0^\infty \frac{f_1(u_1)\ldots f_n(u_n)}{(u_1+\ldots+u_n)^\lambda} du_1 \ldots du_1$$
$$\leq \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \left(\Gamma(2+p_i(\lambda_i-1)) \int_0^\infty u_i^{p_i(1-\lambda_i)-2} \; f_i^{p_i}(u_i) \; du_i \right)^{1/p_i}.$$

Proof. Follows from Theorem 2.1, by putting
$$F_i = f_i, \quad f_i(u_i) = u_i.$$

Remark. By putting
$$\lambda_1 = \frac{\lambda-3+p}{p}, \quad \lambda_2 = \frac{\lambda-3+q}{q}, \quad 1<\lambda<2,$$
we obtain

(1) $\lambda - 1 = \lambda_1 + \lambda_2$.

(2) $2+p(\lambda_1-1) = 2+q(\lambda_2-1) = \lambda-1$, as
$p(\lambda_1-1) = p\lambda_1 - p = \lambda-3 = \lambda_2 q - q = q(\lambda_2-1)$, and
$p(1-\lambda_1) = 3-\lambda < 2$, *as* $\lambda > 1$.

(3) $2+p(\lambda_1-1) < \dfrac{p+\lambda-2}{p} < 1$, $\quad 2+q(\lambda_2-1) < \dfrac{q+\lambda-2}{q} < 1$, which follows since
$$2+p(\lambda_1-1) = 2+\lambda-3 = 1+\lambda-2 < 1+\frac{\lambda-2}{p} = \frac{p+\lambda-2}{p} < 1.$$

From (3), we have, $\Gamma(2+p(\lambda_1-1)) < \Gamma\left(\dfrac{p+\lambda-2}{p}\right)$, and by applying Corollary 2.3

for the special case $\left(n=2 \ \textit{and} \ f(x)=x\right)$, the constant factor is

$$\frac{\Gamma(2+p(\lambda_1-1))}{\Gamma(\lambda)} < \frac{\Gamma^2(2+p(\lambda_1-1))}{\Gamma(\lambda)} < \frac{1}{\Gamma(\lambda)}\Gamma\left(\frac{p+\lambda-2}{p}\right)\Gamma\left(\frac{q+\lambda-2}{q}\right)$$

$$= B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right),$$

provided $\Gamma(2+p(\lambda_1-1))>1$.

This shows that the constant factor in Corollary 2.3 for all cases ($1<\lambda<2$, $x<1$, and $\Gamma(x)>1$) is better than that given in inequality (1.3) (Yang's results [4] and [5]). There are infinitely many numbers satisfying the property ($x<1$ *and* $\Gamma(x)>1$), for example: $\frac{1}{2}<1$ *and* $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}>1$.

References

[1] G. H.Hardy, Note on a theorem of Hilbert concerning series of positive terms, Proc. Math. Soc., 23(2) (1925), Records of Proc. XLV-XLVI.

[2] G.H.Hardy, J.E.Littlewood and G.Polya, Inequalities,Cambridge University Press, Cambridge, UK, 1952.

[3] D.S.Mitrinovic, J.E.Pecaric and A.M.Fink, Inequalities Involving Functions and their Integrals and Derivatives, Kluwer Academic Publisher, Boston, 1991.

[4] B. Yang, On a generalization of Hardy-Hilbert's integrel inequality with a best value Chinese Ann. Math. Ser A 21 (2000) 401-408.

[5] B. Yang, On Hardy-Hilbert's integral inequality, J Math. Anal. Appl. 26 (2001) 295-306.

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