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Common Fixed point theorems for multivalued maps in cone metric spaces

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Abstract

In this paper we prove common fixed point theorems for two multivalued maps in complete cone metric spaces withnormal constant M = 1. Our results generalize and extend the results of Rezapour[12] and others.

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1 Introduction

Nadler Jr.[7] has proved multivalued version of the Banach contraction principle which states that each closed bounded valued contraction map on a complete metric space has a fixed point. Many authors have been using the Hausdroff metric to obtain fixed point results for multivalued maps on metric spaces. Recently Huang and Zhang [5] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Wardowski [13] introduced the concept of multivalued contractions in cone metric spaces and, using the notion of normal cones, obtained fixed point theorems for such mappings. As we know, most of known cones are normal with normal constant M = 1. Further, Rezapour[12] proved two results about common fixed points of multifunctions on cone metric spaces. In this paper we prove common fixed point theorems for two multivalued maps in cone metric spaces with normal constant M = 1 which generalize and extend the results of Rezapour [12] and others.

2 Preliminary Notes

Definition 2.1 *[5]Let E be a real Banach Space and P a subset of E. The set P is called a cone if and only if*
(i) P is closed, non-empty and $P \neq 0$;
(ii)a, b $\in R$, *a, b* ≥ 0, *x, y* $\in P \Rightarrow ax + by \in P$;
(iii) $P \cap (-P) = 0$.
For a given cone $P \subset E$, *we define a partial ordering* $\leq$ *with respect to P by x* $\leq$ *y if and only if y - x* $\in P$. *We write* $x < y$ *to indicate that* $x \rightarrow y$ *but* $x \neq y$, *while* $x << y$ *will stand for y - x* $\in$ *Int P, where Int.P denotes the interior of the set P.*

Definition 2.2 *[5]Let E be a Banach Space and* $P \subset E$ *a cone. The cone P is called normal if there is a number* $M > 0$ *such that for all x, y* $\in E$,
$0 \leq x \leq y$ *implies* $\|x\| \leq M \|y\|$.
The least positive number M satisfying the above inequality is called the normal constant of P.
In the following we always suppose that E is a Banach space, P is a cone in E with Int $P \neq \phi$ *and* $\leq$ *is partial ordering with respect to P.*

Definition 2.3 *[5] Let X be a non-empty set. Suppose that the mapping* $d : X \times X \rightarrow E$ *satisfies:*
(a) $0 < d(x, y)$ *for all x, y* $\in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;
(b) $d(x, y) = d(y, x)$ *for all x, y* $\in X$;
(c) $d(x, y) \leq d(x, z) + d(y, z)$ *for all x, y, z* $\in X$.
Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.4 *[5] Let* $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset R^2, x = R$ *and* $d : X \times X \rightarrow E$ *such that* $d(x, y) = (|x - y|, \alpha |x - y|)$, *where* $\alpha \geq 0$ *is a constant. Then P is a normal cone with normal constant M = 1 and (X, d) is a cone metric space.*

Example 2.5 *Let* $E = l^1, P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0,\ for\ all\ n\}$, *(X,* ρ*) a metric space and d:* $X \times X \rightarrow E$ *defined by* $d(x, y) = \{\frac{\rho(x,y)}{2^n}\}_{n\geq 1}$. *Then P is a normal cone with normal constant M = 1 and (X, d) is a cone metric space.*

This example shows that the category of cone metric spaces is bigger than the metric spaces.

Definition 2.6 *[5] Let (X, d) be a cone metric space. Let* $\{x_n\}$ *be a sequence in X and* $x \in X$.
(i) $\{x_n\}$ *converges to x if for every* $c \in E$ *with* $0 << c$, *there is an* n_0 *such that for all* $n \geq n_0, d(x_n, x) << c$. *We denote this by lim* ${}_{n\rightarrow\infty} x_n = x$ *or* $x_n \rightarrow$

x, $(n \to \infty)$.
(ii) If for any $c \in E$ with with $0 << c$, there is an n_0 such that for all n, $m \geq n_0, d(x_n, x_m) << c$, then $\{x_n\}$ is called a Cauchy sequence in X.
(iii) (X, d) is called a complete cone metric space, if every Cauchy sequence in X is convergent in X.

Lemma 2.7 *[5] Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0 (m, n \to \infty)$.*

Definition 2.8 *[12] Let (X, d) be a cone metric space and $B \subseteq X$.*
(i) A point b in B is called an interior point of B whenever there exists a point p, $0 << p$, such that $N(b, p) \subseteq B$, where $N(b, p) = \{y \in X : d(y, b) << p\}$.
(ii) A subset $A \subseteq X$ is called open if each element of A is an interior point of A.
The family $ß = \{N(x, e) : x \in X, 0 << e\}$ is a sub-basis for a topology on X. We denote this cone topology by τ_c is called Hausdroff and first countable.

Lemma 2.9 *[12] Let (X, d) be a cone metric space, P a normal cone with normal constant M= 1 and A a compact set in (X,τ_c). Then, for every $x \in X$ there exists $a_0 \in A$ such that*

$$\|d(x, a_0)\| = inf_{a \in A} \|d(x, a)\|.$$

Lemma 2.10 *[12] Let (X, d) be a cone metric space, P a normal cone with normal constant M= 1 and A, B two compact sets in (X,τ_c). Then,*

$$sup_{x \in B}\ d'(x, A) < \infty,$$

where $d'(x, A) = inf_{a \in A} \|d(x, a)\|$, for each x in X.

Definition 2.11 *[10] Let (X, d) be a cone metric space, P a normal cone with normal constant $M = 1$, H_c the set of all compact subsets of (x, τ_c) and $A \in H_c(X)$.By using Lemma 2.10, we can define*

$$h_A : H_c(X) \to [0, \infty) \text{ and } d_H : H_c(X) \times H_c(X) \to [0, \infty)$$

by

$$h_A(B) = sup_{x \in A} d'(x, B) \text{ and } d_H(A, B) = max\{h_A(B), h_B(A)\},$$

respectively.

Remark 2.12 *Let (X, d) be a cone metric space with normal constant M = 1. Define* $\rho : X \times X \to [0, \infty)$ *by* $\rho(x, y) = \|d(x,y)\|$*. Then, (X,* ρ*) is a metric space. This implies that for each* $A, B \in H_c$ *and* $x, y \in X$*, we have the following relations*

$$\begin{array}{ll} (i) & d' \leq \|d(x,y)\| + d'(y,A), \\ (ii) & d' \leq d'(x,B) + h_B(A), \text{ and} \\ (iii) & d' \leq \|d(x,y)\| + d'(y,B) + h_B(A). \end{array}$$

3 Main Results

Theorem 3.1 *Let (X, d) be a complete cone metric space with normal constant M = 1 and* $T_1, T_2 : X \to H_c(X)$ *two multivalued maps satisfying the relation*

$$d_H(T_1x, T_2y) \leq a(d'(x, T_1x)) + b(d'(y, T_2y)) + c(d'(x, y))$$

for all $x, y \in X$ *and* $a, b, c \geq 0$*, where* $a + b + c < 1$*. Then* T_1 *and* T_2 *have a common fixed point, that is, there exists* $x \in X$ *such that* $x \in T_1x$ *and* $x \in T_2x$*.*
Proof. Let $x_0 \in X$ *be a arbitray point. By Lemma 2.9, there is* $x_1 \in T_1x_0$ *such that*

$$d'(x_0, T_1x_0) = \|d(x_0, x_1)\|.$$

Also, there is $x_2 \in T_2x_1$ *such that*

$$d'(x_1, T_2x_1) = \|d(x_1, x_2)\|.$$

Thus, we obtain a sequence $\{x_n\}_{n\geq 1}$ *in X such that*

$$x_{2n-1} \in T_1x_{2n-2}, x_{2n} \in T_2x_{2n-1},$$

Therefore

$$d'(x_{2n-2}, T_1x_{2n-2}) = \|d(x_{2n-2}, x_{2n-1})\|,$$

and

$$d'(x_{2n-1}, T_2x_{2n-1}) = \|d(x_{2n-1}, x_{2n})\|,$$

for all $n \geq 1$*. Thus, for all* $n \geq 1$ *we have*

$$\begin{aligned} \|d(x_{2n}, x_{2n+1})\| &= d'(x_{2n}, T_1x_{2n}) \\ &\leq h_{T_2x_{2n-1}}(T_1x_{2n}) \leq d_H(T_2x_{2n-1}, T_1x_{2n}) \end{aligned}$$

$$\begin{aligned}
& \leq ad'(x_{2n}, T_1x_{2n}) + bd'(x_{2n-1}, T_2x_{2n-1}) + cd'(x_{2n}, x_{2n-1}) \\
& = a\,\|d(x_{2n}, x_{2n+1})\| + b\,\|d(x_{2n-1}, x_{2n})\| + c\,\|d(x_{2n}, x_{2n-1})\| \\
or\ (1-a)\,\|d(x_{2n}, x_{2n+1})\| & \leq (b+c)\,\|d(x_{2n-1}, x_{2n})\| \\
or\ \|d(x_{2n}, x_{2n+1})\| & \leq \frac{b+c}{1-a}\,\|d(x_{2n-1}, x_{2n})\| \\
Hence\ \|d(x_{2n}, x_{2n+1})\| & \leq h\ \|d(x_{2n-1}, x_{2n})\|\ \ for\ all\ n \geq 1,
\end{aligned}$$

where $h < 1$ since $a + b + c < 1$.

$$\begin{aligned}
Also\ \|d(x_{2n-1}, x_{2n})\| & = d'(x_{2n-1}, T_2x_{2n-1}) \\
& \leq h_{T_1x_{2n-2}}(T_2x_{2n-1}) \leq d_H(T_1x_{2n-2}, T_2x_{2n-1}) \\
& \leq ad'(x_{2n-2}, T_1x_{2n-2}) + bd'(x_{2n-1}, T_2x_{2n-1}) \\
& \quad + cd'(x_{2n-2}, x_{2n-1}) \\
& = a\,\|d(x_{2n-2}, x_{2n-1})\| + b\,\|d(x_{2n-1}, x_{2n})\| \\
& \quad + c\,\|d(x_{2n-2}, x_{2n-1})\|\ \ or \\
(1-b)\,\|d(x_{2n-1}, x_{2n})\| & \leq (a+c)\,\|d(x_{2n-2}, x_{2n-1})\|\ \ or \\
\|d(x_{2n-1}, x_{2n})\| & \leq \frac{a+c}{1-b}\,\|d(x_{2n-2}, x_{2n-1})\| \\
Hence\ \|d(x_{2n-1}, x_{2n})\| & \leq h\ \|d(x_{2n-2}, x_{2n-1})\|\ \ for\ all\ n \geq 1,
\end{aligned}$$

where $h < 1$ since $a + b + c < 1$.

This implies that $\|d(x_m, x_{m+1})\| \leq \|d(x_{m-1}, x_m)\|$, *for all* $m \geq 1$.
Then, for $n > m$, *we have*

$$\begin{aligned}
\|d(x_n, x_m)\| & \leq \textstyle\sum_{i=m+1}^{n} \|d(x_i, x_{i-1})\| \\
& \leq (h^{n-1} + \ \dots\ + h^m)\,\|d(x_0, x_1)\| \leq \frac{h^m}{1-h}\,\|d(x_0, x_1)\|
\end{aligned}$$

This implies that

$$lim_{m,n\to\infty}\,\|d(x_n, x_m)\| = 0.$$

By Lemma 2.7, $\{x_n\}_{n\geq 1}$ *is a Cauchy sequence in X. Thus, there exists* $x^* \in X$ *such that* $x_n \to x^*$. *Now by Using Remark 2.12, we have*

$$\begin{aligned}
d'(x^*, T_1x^*) & \leq d'(x^*, T_2x_{2n-1}) + h_{T_2x_{2n-1}}(T_1x^*) \\
& \leq d'(x^*, T_2x_{2n-1}) + d_H(T_2x_{2n-1}, T_1x^*) \\
& \leq \|d(x^*, x_{2n})\| + ad'(x^*, T_1x^*) + bd'(x_{2n-1}, T_2x_{2n-1}) + cd'(x^*, x_{2n-1}),
\end{aligned}$$

for all $n \geq 1$. Hence

$$\begin{aligned}
d'(x^*, T_1x^*) & \leq \frac{1}{1-a}\,\|d(x^*, x_{2n})\| + \frac{b}{1-a}d'(x_{2n-1}, T_2x_{2n-1}) + \frac{c}{1-a}d'(x^*, x_{2n-1}), \\
& = \frac{1}{1-a}\,\|d(x^*, x_{2n})\| + \frac{b}{1-a}\,\|d(x_{2n-1}, x_{2n})\| + \frac{c}{1-a}\,\|d(x^*, x_{2n-1})\|,
\end{aligned}$$

for all $n \geq 1$. Therefore, $d'(x^*, T_1x^*) = 0$. By Lemma 2.9 $x^* \in T_1x^*$.
On the other hand, similarly we have

$$\begin{aligned} d'(x^*, T_2x^*) &\leq d'(x^*, T_1x_{2n}) + h_{T_1x_{2n}}(T_2x^*) \\ &\leq d'(x^*, T_1x_{2n}) + d_H(T_1x_{2n}, T_2x^*) \\ &\leq \|d(x^*, x_{2n+1})\| + ad'(x_{2n}, T_1x_{2n}) + bd'(x^*, T_2x^*) + cd'(x_{2n}, x^*), \end{aligned}$$

for all $n \geq 1$. Hence

$$\begin{aligned} d'(x^*, T_2x^*) &\leq \frac{1}{1-b}\|d(x^*, x_{2n+1})\| + \frac{a}{1-b}d'(x_{2n}, T_1x_{2n}) + \frac{c}{1-b}d'(x_{2n}, x^*), \\ &= \frac{1}{1-b}\|d(x^*, x_{2n+1})\| + \frac{a}{1-b}\|d(x_{2n}, x_{2n+1})\| + \frac{c}{1-b}\|d(x_{2n}, x^*)\|, \end{aligned}$$

for all $n \geq 1$. Therefore, $d'(x^*, T_2x^*) = 0$. By Lemma 2.9 $x^* \in T_2x^*$. Thus, x^* is a common fixed point of T_1 and T_2.

Theorem 3.2 *Let (X, d) be a complete cone metric space with normal constant $M = 1$ and $T_1, T_2 : X \to H_c(X)$ two multivalued maps satisfying the relation*

$$d_H(T_1x, T_2y) \leq a(d'(x, T_2y)) + b(d'(y, T_1x)) + c(d'(x, y))$$

for all x, $y \in X$ and a, b, $c \geq 0$, where $a + b + c < 1$. Then T_1 and T_2 have a common fixed point.
Proof. A similar argument to that of the proof of Theorem 3.1 shows that there exists a Cauchy sequence $\{x_n\}_{n\geq 1}$ in X such that

$$x_{2n-1} \in T_1x_{2n-2}, x_{2n} \in T_2x_{2n-1},$$

Therefore

$$d'(x_{2n-2}, T_1x_{2n-2}) = \|d(x_{2n-2}, x_{2n-1})\|,$$

and

$$d'(x_{2n-1}, T_2x_{2n-1}) = \|d(x_{2n-1}, x_{2n})\|,$$

for all $n \geq 1$. Thus, there exists $x^* \in X$ such that $x_n \to x^*$. Now by using Remark 2.12, we have

$$\begin{aligned} d'(x^*, T_1x^*) &\leq d'(x^*, T_2x_{2n-1}) + h_{T_2x_{2n-1}}(T_1x^*) \\ &\leq d'(x^*, T_2x_{2n-1}) + d_H(T_2x_{2n-1}, T_1x^*) \\ &\leq \|d(x^*, x_{2n})\| + ad'(x^*, T_2x_{2n-1}) + bd'(x_{2n-1}, T_1x^*) + cd'(x^*, x_{2n-1}), \\ &\leq \|d(x^*, x_{2n})\| + a\|d(x^*, x_{2n})\| + bd'(x^*, T_1x^*) + bd'(x^*, x_{2n-1}) \\ &\quad + cd'(x^*, x_{2n-1}), \end{aligned}$$

for all $n \geq 1$. Hence

$$d'(x^*, T_1x^*) \leq \frac{1+a}{1-b}\|d(x^*, x_{2n})\| + \frac{b+c}{1-b}\|d(x^*, x_{2n-1})\|,$$

for all $n \geq 1$. *Therefore,* $d'(x^*, T_1x^*) = 0$. *By Lemma 2.9* $x^* \in T_1x^*$.
Also, we have

$$\begin{aligned} d'(x^*, T_2x^*) &\leq d'(x^*, T_1x_{2n}) + h_{T_1x_{2n}}(T_2x^*) \\ &\leq d'(x^*, T_1x_{2n}) + d_H(T_1x_{2n}, T_2x^*) \\ &\leq \|d(x^*, x_{2n+1})\| + ad'(x_{2n}, T_2x^*) + bd'(x^*, T_1x_{2n}) + cd'(x_{2n}, x^*), \\ &\leq \|d(x^*, x_{2n+1})\| + ad'(x^*, T_2x^*) + ad'(x^*, x_{2n}) + b\,\|d(x^*, x_{2n+1})\| \\ &\quad +cd'(x^*, x_{2n}), \end{aligned}$$

$for\ all\ n \geq 1. Hence$

$$d'(x^*, T_2x^*) \leq \frac{1+b}{1-a}\,\|d(x^*, x_{2n+1})\| + \frac{a+c}{1-a}\,\|d(x^*, x_{2n})\|\,,$$

for all $n \geq 1$. *Therefore,* $d'(x^*, T_2x^*) = 0$. *By Lemma 2.9* $x^* \in T_2x^*$. *Thus,* x^* *is a common fixed point of* T_1 *and* T_2.

Remark 3.3 *If we take* $a = b$ *and* $c = 0$ *in Theorem 3.1, then we get the Theorem 2.3 of Rezapour[12].*

Remark 3.4 *If we take* $a = b$ *and* $c = 0$ *in Theorem 3.2, then we get the Theorem 2.4 of Rezapour[12].*

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