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Bernstein Direct Method for Solving Variational Problems

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Abstract

A simple algorithm for solving variational problems via Bernstein orthonormal polynomials of degree six is proposed. The sixth order Bernstein polynomials are orthonormalized and their operational matrix of integration is derived. Using this operational matrix of integration, the variational problems are reduced to the solution of systems of algebraic equations. The method of Lagrange multipliers is used to solve illustrative problems with free and constrained boundaries.

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1 Introduction

Many problems of mathematical physics are connected with the calculus of variations, one of the very central fields of analysis. Some of its uses in both physics and mathematics are as follows.

• Unification of diverse areas of physics-using energy as a key concept.

•• Starting point for new, complex areas of physics and engineering. In general relativity the geodesic is taken as the minimum path of light pulse or the free fall path of a particle in curved Riemannian space. Variational principles have been applied extensively in modern control theory and they also appear in quantum field theory.

••• Variational analysis provides a proof of the completeness of the Sturm-Liouville eigenfunctions and establishes a lower bound for the eigenvalues. Similar results follow for the eigenvalues and eigenfunctions of the Hilbert-Schmidt integral equation.

•••• Mathematical models of several problems in the system and control theory, modern mechanics, economics may be reduced to systems of differential, differential-integral or integral equations as boundary-initial value problems. These equations are nonlinear in general and hence analytical solutions are difficult to obtain in nontrivial cases. Variational principles provide an alternative to search for analytical solutions of such problems. The idea is to formulate variational functionals whose stationarity conditions lead to equations that describe the problems. The Euler-Lagrange equations obtained by applying the well known procedure in the calculus of variation [4], usually leads to equations that are difficult to solve. Many authors [2,3,6,8] have tried various transform methods to overcome these difficulties in the problem of extremization of functional systems. The main idea of a direct method for solving variational problems is to convert the problem of extremization of a functional into one involving a finite number of variables. The Ritz method [4], usually based on the subspaces of kinematically admissible complete functions, is the most commonly used approach in direct methods of solving variational problems.

Orthogonal functions are special functions in the space of which approximate solutions of variational problems are sought. Using the Ritz method and orthogonal functions with their operational matrices of integration, one can reduce a variational problem to systems of algebraic equations. Several orthogonal functions and wavelets such as Walsh functions [3, 5], Laguerre [8], Legendre [2], Chebyshev polynomials [6], Fourier series [10], Haar wavelets [7], Legendre wavelets [11] were used to solve variational problems. Babolian et. al. [1], Razzaghi et. al. [9] used triangular orthogonal functions and rationalized Haar functions respectively for the purpose.

In this paper, we solve variational problems using Bernstein polynomials of degree six. First, some simple properties of Bernstein polynomials are given, and then these seven polynomials of degree six are used to construct a family of seven orthonormal polynomials of the same degree which are named as Bernstein orthonormal polynomials. The operational matrix of integration is derived and a direct method for solving variational problems is presented. The method is based on (i) assuming representations of admissible functions by orthonormal Bernstein polynomials with coefficients to be determined; (ii) using the operational matrix of integration for performing integration; (iii) finding the necessary condition for extremization; and (iv) solving for the algebraic equations obtained from the previous steps to evaluate Bernstein coefficients.

The above four steps are the standard ones followed by the other authors [2,3,5,6,7,8,10], as well. To the authors' knowledge, the Bernstein operational matrix has not been derived before and it is for the first time this matrix has been derived and is being used to solve the variational problems.

The calculation procedures for Laguerre polynomials [8], Legendre polynomials [2], and Chebyshev polynomials [6] are usually too tedious, and some recursive formulae are still waiting for developments [7]. In [7,1], authors had derived Haar product matrix and triangular functions (TF) product matrix, respectively to solve the variational problems. And in [1], the order of matrix is too large, of the order 512, to achieve accuracy comparable to the ones we obtain by using matrix of order 5. Hsiao [7] used the product matrix of order 8 but the error is quite large compared to our method, proposed in this paper. Also, the low order of the matrix makes the computations very simple in this paper.

2 The Bernstein polynomials

A Bernstein polynomial, named after Sergei Natanovich Bernstein, is a polynomial in the Bernstein form that is a linear combination of Bernstein basis polynomials.
The Bernstein basis polynomials of degree n are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad , \qquad for\ \ i = 0,1,2,\cdots,n\ . \tag{1}$$

There are $(n+1)$ n^{th} degree Bernstein basis polynomials and form a basis for the linear space V_n consisting of all polynomials of degree less than or equal to n in **R[x]-**the integral domain of polynomials over the real field **R**. For mathematical convenience, we usually set $B_{i,n} = 0$ if $i < 0$ or $i > n$.

Any polynomial $B(x)$ of degree n in **R[x]** may be written as

$$B(x) = \sum_{i=0}^{n} \beta_i B_{i,n}(x). \tag{2}$$

Then $B(x)$ is called a polynomial in Bernstein form or Bernstein polynomial of degree n. The coefficients β_i are called Bernstein or Bezier coefficients. But several mathematicians call Bernstein basis polynomials $B_{i,n}(x)$ as the Bernstein polynomials. We will follow this convention as well. These polynomials have the following properties:

(i) $B_{i,n}(0) = \delta_{i0}$ and $B_{i,n}(1) = \delta_{in}$, where δ is the Kronecker delta function.

(ii) $B_{i,n}(t)$ has one root, each of multiplicity i and $n-i$, at $t = 0$ and $t = 1$ respectively.

(iii) $B_{i,n}(t) \ge 0$ for $t \in [0,1]$ and $B_{i,n}(1-t) = B_{n-i,n}(t)$.

(iv) For $i \ne 0$, $B_{i,n}$ has a unique local maximum in $[0,1]$ at $t = i/n$ and the maximum value $i^i n^{-n}(n-i)^{n-i}\binom{n}{i}$.

(v) The Bernstein polynomials form a partition of unity i.e. $\sum_{i=0}^{n} B_{i,n}(t) = 1$.

(vi) It has a degree raising property in the sense that any of the lower-degree polynomials (degree $< n$) can be expressed as linear combinations of polynomials of degree n. We have,

$$B_{i,n-1}(t) = \left(\frac{n-i}{n}\right) B_{i,n}(t) + \left(\frac{i+1}{n}\right) B_{i+1,n}(t).$$

(vii) Let $f(x) \in C\,[0,1]-$ (the class of continuous functions on $[0,1]$), then

$$B_n(f)(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_{i,n}(x) \text{ converges to } f(x) \text{ uniformly on } [0,1] \text{ as } n \to \infty.$$

$(viii)$ Let $f(x) \in C^{(k)}[0,1]-$ (the class of $k-$ times differentiable function with $f^{(k)}$ continuous), then

$\left\|B_n(f)^{(k)}\right\|_\infty \le \frac{(n)_k}{n^k}\left\|f^{(k)}\right\|_\infty$ and $\left\|f^{(k)} - B_n(f)^{(k)}\right\|_\infty \to 0$ as $k \to \infty$, where $\|.\|_\infty$ is the sup. norm and $\frac{(n)_k}{n^k} = \left(1-\frac{0}{n}\right)\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$ is an eigenvalue of B_n; the corresponding eigenfunction is a polynomial of degree k.

3 The orthonormal polynomials

Using Gram- Schmidt orthonormalization process on $B_{i,n}$ and normalizing, we obtain a class of orthonormal polynomials from Bernstein polynomials. We call them orthonormal Bernstein polynomials of order n and denote them by $b_{on}, b_{1n}, \cdots, b_{nn}$ for $n = 6$ the orthonormal polynomials are given by

$$b_{06}(t) = \sqrt{13}(-1+t)^6 \tag{3}$$

$$b_{16}(t) = -\sqrt{11}(-1+t)^5(-1+13t)$$

$$b_{26}(t) = 3(-1+t)^4(1-24t+78t^2)$$

$$b_{36}(t) = -\sqrt{7}(-1+t)^3(-1+33t-198t^2+286t^3)$$

$$b_{46}(t) = \sqrt{5}(-1+t)^2(1-40t+330t^2-880t^3+715t^4)$$

$$b_{56}(t) = \sqrt{3}(-1+46t-495t^2+2100t^3-4125t^4+3762t^5-1287t^6)$$

$$b_{66}(t) = (1-48t+540t^2-2400t^3+4950t^4-4752t^5+1716t^6)$$

A function $f \in L^2[0,1]$ may be written as

$$f(t) = \lim_{n\to\infty} \sum_{i=0}^{n} c_{in}\, b_{in}(t), \tag{4}$$

where, $c_{in} = \langle f, b_{in} \rangle$ and $\langle , \rangle$ is the standard inner product on $L^2[0,1]$. If the series (4) is truncated at $n = m$, we get an approximation $\tilde{f}$ of f as,

$$f \cong \tilde{f} = \sum_{i=0}^{m} c_{im}\, b_{im} = C^T B(t), \tag{5}$$

where,
$$C = [c_{0m}, c_{1m}, \cdots, c_{mm}]^T \tag{6}$$

and
$$B(t) = [b_{0m}(t), b_{1m}(t), \cdots, b_{mm}(t)]^T. \tag{7}$$

4 The operational matrix of integration

In this section, the Bernstein operational matrix of integration is derived. So, let us start with the integral property of the basic operational matrix

$$\int_a^t \cdots \int_a^t \varphi(\sigma)(d\sigma)^k \approx P_{m+1}^k \varphi(t), \tag{8}$$

where $\varphi(t) = [\varphi_0(t), \varphi_1(t), \ldots, \varphi_m(t)]^T$ in which the elements $\varphi_0(t), \varphi_1(t), \ldots, \varphi_m(t)$ are the basis functions, orthogonal on a certain interval $[a,b]$ and P is the operational matrix for integration of $\varphi(t)$. Note that P is a constant matrix of order $(m+1)\times(m+1)$.
The orthonormal Bernstein polynomials operational matrix of integration of order $m \times m$ will be derived now. To achieve this, consider the following integral

$$\int_0^t b_{i\,m}(x)dx = \varphi_i(t), \qquad 0 \le t < 1, \qquad i = 0,1,\cdots,m.$$

$$= \sum_{j=0}^{m} c_{j\,m}^{i}\, b_{j\,m}(t),$$

$$= [c_{0\,m}^{i}, c_{1m}^{i}, \cdots, c_{m\,m}^{i}]\, B(t). \tag{9}$$

Using equations (7) and (9), we obtain

$$\int_0^t B(x)\,dx = P_{m+1}\, B(t), \tag{10}$$

where the operational matrix P_{m+1} of integration associated with orthonormal Bernstein polynomials is given by

$$P_{m+1} = (c_{j\,m}^{i})_{i,\ j=0}^{m}$$

and

$$c_{j\,m}^{i} = \left\langle \varphi_i, b_{j\,m} \right\rangle .$$

For $m = 6$, the operational matrix P_7 (denoted as p) is given as:

$$\text{p} := \begin{pmatrix} 0.132653 & 0.253433 & 0.219333 & 0.195022 & 0.164421 & 0.127498 & 0.0735612 \\ -0.00938639 & 0.112245 & 0.21998 & 0.175011 & 0.152727 & 0.116644 & 0.0679426 \\ 0.00141505 & -0.0169216 & 0.0918367 & 0.184074 & 0.129435 & 0.109257 & 0.059833 \\ -0.000340352 & 0.00407002 & -0.0220888 & 0.0714286 & 0.144884 & 0.0831306 & 0.0584945 \\ 0.00011506 & -0.00137592 & 0.00746739 & -0.0241473 & 0.0510204 & 0.100996 & 0.0361269 \\ -0.000049514 & 0.000592101 & -0.00321345 & 0.0103913 & -0.0219557 & 0.0306122 & 0.0486035 \\ 0.0000214402 & -0.000256387 & 0.00139147 & -0.00449958 & 0.00950709 & -0.0132555 & 0.0102041 \end{pmatrix}$$

As,

$$\int_0^1 B(t)\, B^T(t)\, dt = I\,;$$

where I is 7×7 identity matrix, it makes the subsequent computations very simple compared to the methods of Hsio [7], Babolian et. al. [1] and Razzaghi et.al. [9].

5 The Bernstein direct method

Consider the problem of finding the extremum of the functional

$$J(x) = \int_0^1 F\big[t, x(t), \dot{x}(t)\big] dt\,. \tag{11}$$

The necessary condition for $x(t)$ to extremize $J(x)$ is that it should satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$$

with appropriate boundary conditions. However, the above differential equation can be integrated easily only for simple cases. Thus numerical and direct methods such as the well-known Ritz and Galerkin methods [4], have been developed to solve variational problems. In this paper the Bernstein orthonormal polynomials are used to establish the direct method for variational problems.

Suppose, the rate variable $\dot{x}(t)$ can be expressed approximately as

$$\dot{x}(t) \cong \tilde{\dot{x}}(t) = \sum_{i=0}^{m} c_{i\,m}\, b_{i\,m} = C^T B(t)\,. \tag{12}$$

Integrating equation (12) from 0 to t and using (10), we represent $x(t)$ as

$$\begin{aligned} x(t) &= \int_0^t x(t')\, dt' + x(0)\;, \\ &\approx C^T P_{m+1}\, B(t) + x(0) = \tilde{x}(t)\,. \end{aligned} \tag{13}$$

We can also express t in terms of $B(t)$ as

$$t \cong d^T B(t)\,, \text{ where } d^T = \big[d_{0\,m},\, d_{1m}\,, \cdots,\, d_{m\,m}\big]. \tag{14}$$

The other terms in the functional of equation (11) are known functions of the independent variable t and can be expanded into Bernstein orthonormal polynomials via equation (14). Substituting Eqs. (12)- (14) in Eq. (11) the functional $J(x)$ becomes a function of c_i, $i = 0,1,\cdots, m$, and we finally have

$$J = J(c_0, c_1, \ldots, c_m). \tag{15}$$

The original extremization of the functional problem (11) becomes the extremization of a function of a finite set of variables in Eq.(15).
Hence, to find the extremum of $J(x)$, we take partial derivatives of J with respect to c_i and set them equal to zero. Thus, we obtain

$$\frac{\partial J}{\partial c_i} = 0, \qquad i = 0,1,\cdots, m. \tag{16}$$

Solving for c_i, and substituting in Eq.(13), we get the solution.
We establish the detailed procedure via several classical problems.

6 Numerical examples

6.1. First order functional extremal with two fixed boundary conditions

Consider the problem of finding the extremal of the functional [7]

$$J(x) = \int_0^1 [\dot{x}^2(t) + t\,\dot{x}(t)]\,dt. \tag{17}$$

The boundary conditions are the initial and the final conditions

$$x(0) = 0, \tag{18}$$

$$x(1) = \frac{1}{4}. \tag{19}$$

To solve this problem by the proposed method, we assume $\dot{x}(t)$ can be expanded in terms of the Bernstein orthonormal polynomials as in Eq.(12). We solve this and the subsequent examples by taking $m = 6$ and note that a very high degree of accuracy is achieved.
Substituting Eqs. (12) and (14) into Eq. (17), we have

$$J \approx \int_0^1 \left[C^T B(t) B^T(t) C + C^T B(t) B^T(t) d \right] dt \,. \tag{20}$$

As,
$$\int_0^1 B(t) B^T(t) d(t) = I \,, \tag{21}$$

$$J \approx C^T C + C^T d \,. \tag{22}$$

Substituting the initial boundary conditions (18) into (13), we get

$$x(t) \approx C^T P B(t) \,. \tag{23}$$

Hence the final boundary condition (19) substituted in (23) yields

$$x(1) = C^T P B(1) = \frac{1}{4} \,. \tag{24}$$

We minimize (22) subject to (24) using Lagrange multiplier. Suppose

$$\tilde{J} = J + \lambda \left[C^T P B(1) - \frac{1}{4} \right]$$

$$= C^T C + C^T d + \lambda \left[C^T P B(1) - \frac{1}{4} \right], \tag{25}$$

where λ is Lagrange multiplier. To minimize, the partial derivatives of $\tilde{J}$ with respect to C^T and λ are taken and set to zero,

$$\frac{\partial \tilde{J}}{\partial C^T} = 0, \qquad \frac{\partial \tilde{J}}{\partial \lambda} = 0 \,. \tag{26}$$

Thus,

$$2C + d + \lambda P B(1) = 0 \,, \tag{27}$$

and

$$C^T P B(1) = \frac{1}{4} \,. \tag{28}$$

Eqs.(27) and (28) consist of six simultaneous linear equations which are used to determine $c_{06}, c_{16}, \cdots, c_{66}$ and λ.

Therefore,

$$\tilde{x}(t) = [0.225347,\ 0.148064,\ 0.089286,\ 0.047246,\ 0.019965, 0.005155, 0]\,B(t)\,, \tag{29}$$

and

$$\tilde{x}(t) = [0.028615, 0.072387, 0.089286, 0.089242, 0.078972, 0.061859, 0.035714]\,B(t) \tag{30}$$

If the Euler equation is used to solve the above problem, the exact answer is obtained as

$$\dot{x}(t) = \frac{1}{2}(1-t)\,, \tag{31}$$

$$x(t) = \frac{t}{2}(1-\frac{t}{2})\,. \tag{32}$$

$$\text{Let}\, E_2 = \left\| x(t) - \tilde{x}(t) \right\|_2 = \left[\int_0^1 \left| x(t) - \tilde{x}(t) \right|^2 dt \right]^{1/2}\,, \tag{33}$$

where $x(t)$ is the exact solution (32) and $\tilde{x}(t)$ is the approximate solution (30), and $\left\| \bullet \right\|_2$ is the L^2-norm.

$$\text{Then, } E_2 = 2.19 \times 10^{-7}\,. \tag{34}$$

Whereas, when triangular function method [1] is applied, we have

$$E_2^8 = 7.132 \times 10^{-4}\ ,\quad E_2^{256} = 7 \times 10^{-7} \text{ and } E_2^{512} = 1.0 \times 10^{-7}\,. \tag{35}$$

In this and the subsequent examples E_2^m will denote the error in L^2-norm when approximation is performed by using $m \times m$ matrix in triangular function method.

Figs. 1, 3, 5 and 7 compare the exact solutions $x(t)$ with the approximate solutions $\tilde{x}(t)$ (denoted by $x1(t)$ in the Figs.) and Figs. 2, 4, 6 and 8 show the errors $E(t) = \tilde{x}(t) - x(t)$. Tables are given at end comparing our results with that of other authors [7, 1, 9] and exact solutions.

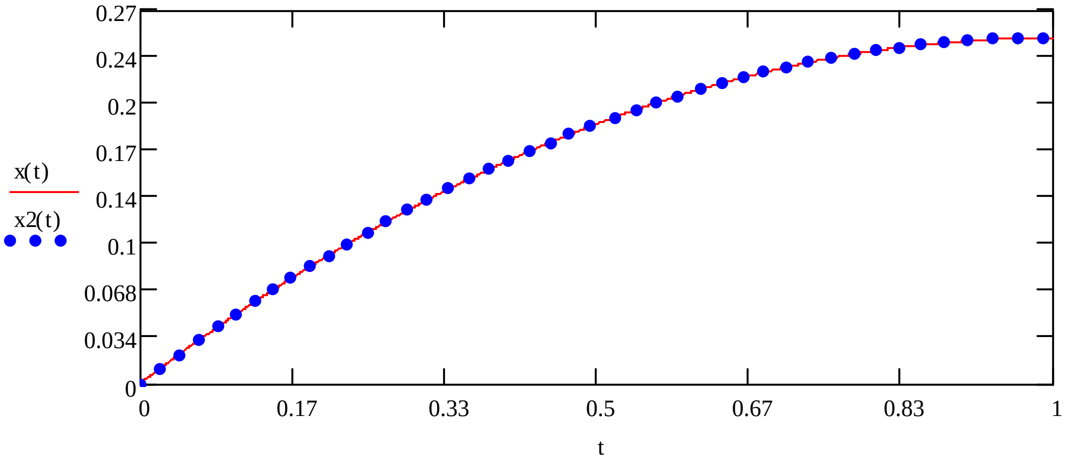

Fig.1. Compares the exact solution with the approximate solution.

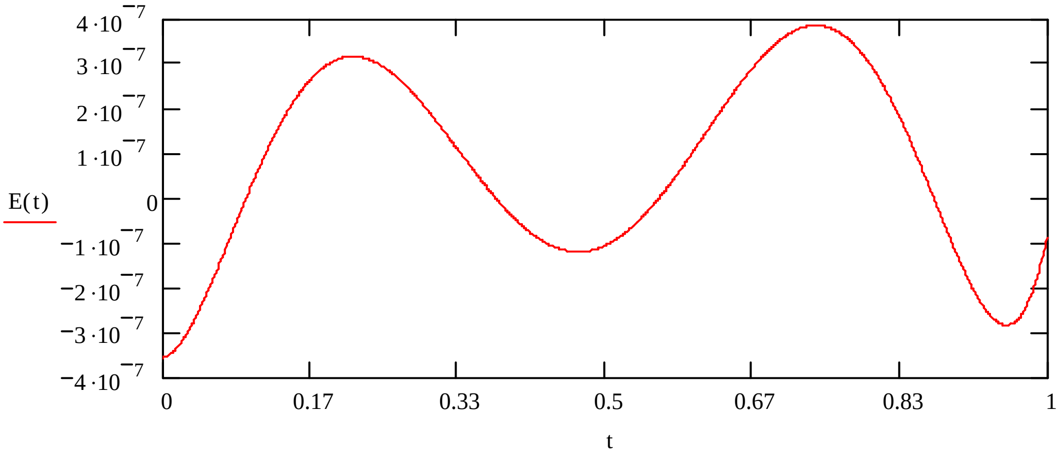

Fig.2. Shows the error between them.

6.2. First order functional extremal with a fixed and a moving boundary conditions

We consider the same functional extremal of Eq.(17) but with unspecified $x(1)$, namely

$$x(0) = 0\,, \tag{36}$$

$$x(1) = \textit{unspecified}\,. \tag{37}$$

Another condition may be found from $F[t,\ x(t),\ \dot{x}(t)]$.

$$F_{\dot{x}}\Big|_{t=1} = 0, \qquad \dot{x}(1) = -\frac{1}{2}. \qquad [8] \tag{38}$$

The condition $\dot{x}(1) = -\frac{1}{2}$ along with $\dot{x}(t) = C^T B(t) \Rightarrow C^T B(1) = -\frac{1}{2}$,
follows from (3). (39)
Substituting Eqs.(12), (13), (14) and (36) into Eq.(17), we get

$$J \approx C^T C + C^T d \quad . \tag{40}$$

J is extremized subject to the constraint (38). Let λ be the Lagrange multiplier, so that, we define $\tilde{J}$ as

$$\tilde{J} \approx J + \lambda\left[C^T B(1) + \frac{1}{2}\right]. \tag{41}$$

Equating partial derivatives of $\tilde{J}$ with respect to C^T and λ equal to zero, we have

$$2C + d + \lambda B(1) = 0, \tag{42}$$

and

$$C^T B(1) + \frac{1}{2} = 0 \Rightarrow C_{66} = -\frac{1}{14}. \tag{43}$$

Solving (42) subject to (43), we get

$$C^T = [-0.032192, -0.088838, -0.125, -0.141737, -0.139754, -0.118563, -0.071429], \tag{44}$$

Giving the approximate solution $\tilde{x}(t)$ as

$$\tilde{x}(t) = [-0.003577, -0.016452, -0.035714, -0.052495, -0.060782, -0.056704, -0.035714]B(t) \tag{45}$$

and

$\tilde{\dot{x}}(t) = C^T B(t)$, where C^T is given by (44).

Analytic solution via Euler's equation is

$$\dot{x}(t) = -\frac{t}{2}, \tag{46}$$

$$x(t) = -\frac{t^2}{4}. \tag{47}$$

The least squares error E_2 for this problem is

$$E_2 = 1.5 \times 10^{-7}, \tag{48}$$

which is for superior than the errors E_2^m obtained in [1]. We quote three values of E_2^m for $m = 8,\ 256$ and 512.

$$E_2^8 = 7.132 \times 10^{-4}, \quad E_2^{256} = 7 \times 10^{-7} \text{ and } E_2^{512} = 10^{-7}. \tag{49}$$

The comparison of the solutions via Euler's analytic method and via Bernstein's direct method is shown in Fig.3 and the corresponding error in Fig.4.

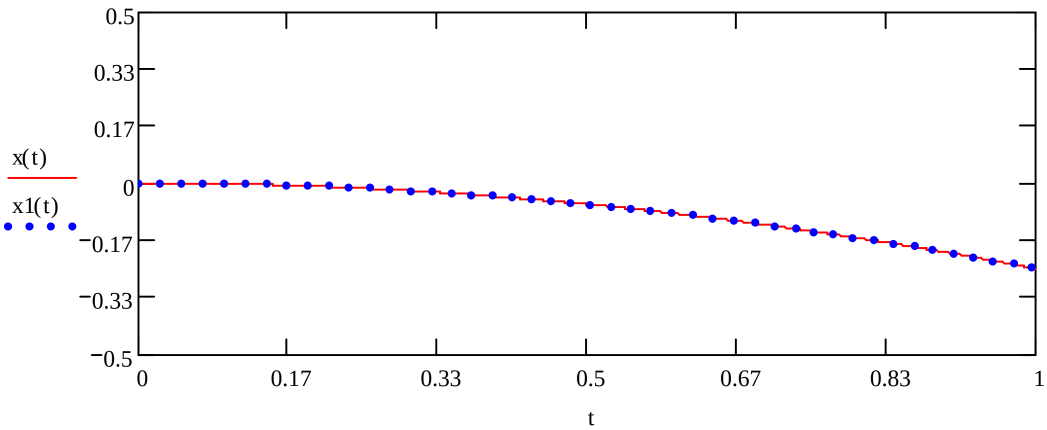

Fig.3. Compares the exact solution with the approximate solution**.**

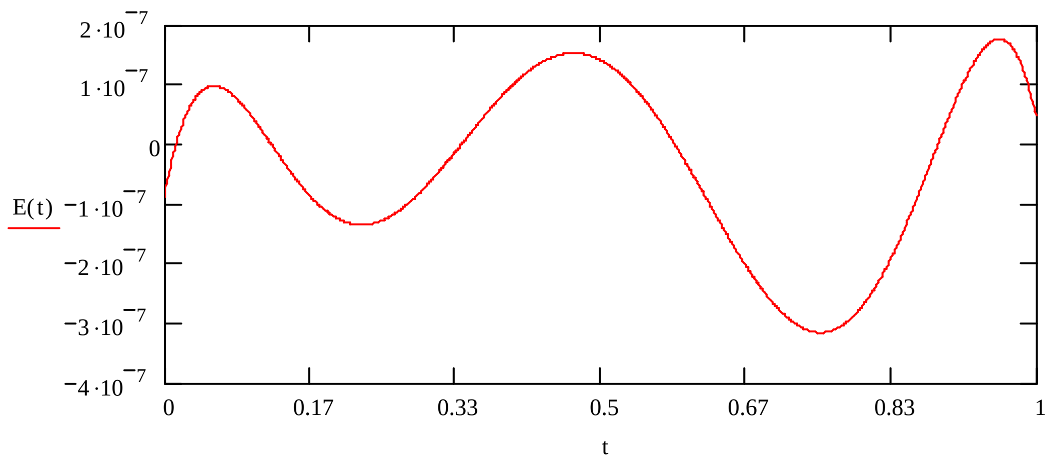

Fig.4. Shows the error between them.

6.3. First order functional extremal with two fixed boundary conditions

Let us consider the problem of searching for the extremum of this functional [2, 5, 11]

$$J = \int_0^1 [\dot{x}^2(t) + t\,\dot{x}(t) + x^2(t)]\,dt\,, \tag{50}$$

with the following boundary conditions;

$$x(0) = 0\,, \qquad x(1) = \frac{1}{4} \tag{51}$$

The exact solution of this problem is

$$x(t) = \frac{-e^{-t}\,[(-1+e^{t})(e-2e^{2}-2e^{t}+e^{1+t})]}{4(-1+e^{2})} \tag{52}$$

Arguments similar to the previous problems lead to

$$J = C^T C + C^T d + C^T P\,P^T C\,, \tag{53}$$

$$\tilde{J} = J + \lambda\left[C^T P\,B(1) - \frac{1}{4}\right]. \tag{54}$$

And

$$\tilde{\dot{x}}(t) = [0.199237, 0.133942, 0.090838, 0.061513, 0.041205, 0.026628, 0.013886]\,B(t)\,, \tag{55}$$

$$\tilde{x}(t) = [0.025283, 0.064196, 0.080389, 0.082631, 0.075535, 0.060857, 0.035714]\,B(t)\,. \tag{56}$$

The least squares error E_2 is given as

$$E_2 = 2.1 \times 10^{-7} \tag{57}$$

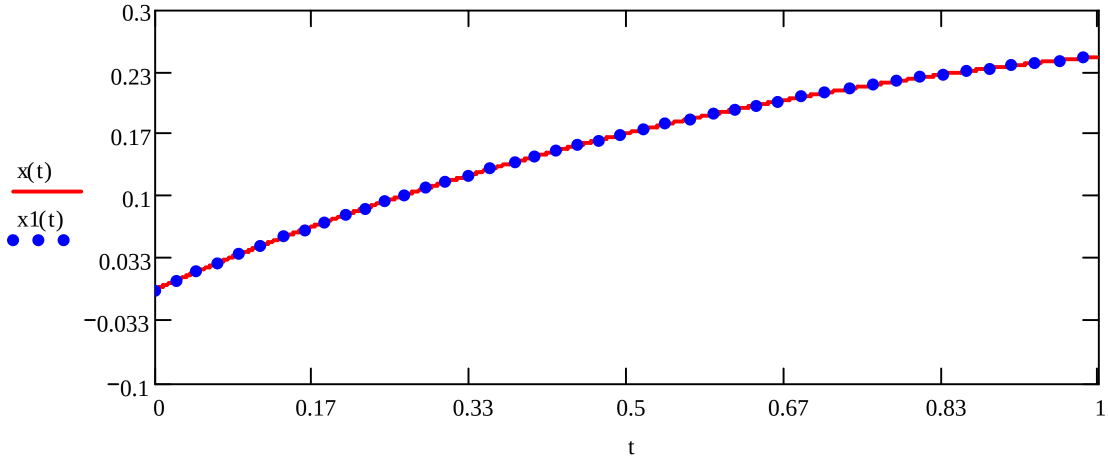

Fig.5. Compares the exact solution with the approximate solution for example 6.3.

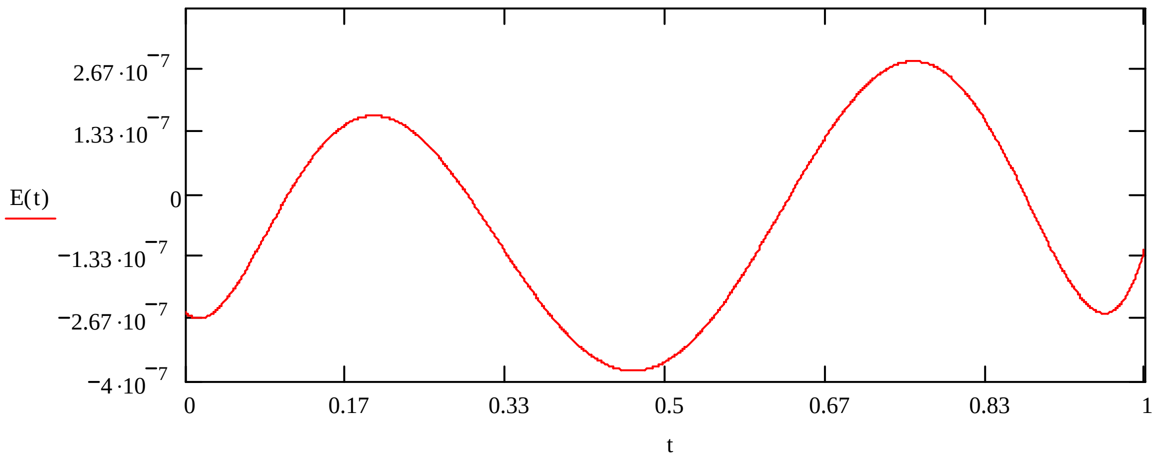

Fig.6. Shows the error between them for example 6.

6.4. Second-order functional extremal with two fixed and two moving boundary conditions

Consider the following functional extremal problem

$$J = \int_0^1 [\frac{1}{2}\ddot{x}^2(t) + 4(1-t)\dot{x}(t)]dt \ , \tag{58}$$

with

$$x(0) = 0 \tag{59}$$

$$\dot{x}(0) = 0 \tag{60}$$

and $x(1)$, $\dot{x}(1)$ unspecified. (61)

The natural boundary conditions are found from [7]

$$F_{\dot{x}} - \frac{d}{dt}(F_{\ddot{x}})\Big|_{t=1} = 0 \Rightarrow 4(1-t) - \dddot{x}\Big|_{t=1} = 0 \Rightarrow \dddot{x}(1) = 0 \tag{62}$$

$$F_{\ddot{x}}\Big|_{t=1} = 0 \Rightarrow \ddot{x}(1) = 0 . \tag{63}$$

Expanding $\dddot{x}(t)$ into Bernstein orthonormal polynomial series and truncating it at $m = 6$, we get

$$\tilde{\dddot{x}}(t) = \sum_{i=0}^{6} C_{i6}\, b_{i6} = C^T B(t) . \tag{64}$$

Integrating (64) repeatedly twice gives

$$\tilde{\ddot{x}}(t) = C^T P\, B(t) + \ddot{x}(0) , \tag{65}$$

$$\tilde{\dot{x}}(t) = C^T P^2\, B(t) + \ddot{x}(0)\, t \text{, as } \dot{x}(0) = 0 , \tag{66}$$

and

$$\tilde{x}(t) = C^T P^3\, B(t) + \ddot{x}(0)\, d^T P\, B(t) ,\ x(0) = 0 \tag{67}$$

where $t = d^T B(t)$, from Eqs.(65) and (63), we get

$$\ddot{x}(0) = -C^T PB(1) . \tag{68}$$

Expressing $4 - 4t = f^T B(t)$, and using Eqs. (65) to (68), we get

$$\begin{aligned} J = \frac{1}{2} C^T P\, P^T C - C^T P^2 B(1)\, B^T(1)\, P^T C + \\ \frac{1}{2} C^T P\, B(1)\, B^T(1)\, P^T C + f^T \left(P^2\right)^T C - C^T P\, B(1)\, d^T f \end{aligned} \tag{69}$$

Differentiating J with respect to C^T and equating it to zero, we have

$$\begin{aligned} \frac{\partial J}{\partial C^T} = P\, P^T C - 2P^2 B(1)\, B^T(1)\, P^T C + \\ P\, B(1)\, B^T(1)\, P^T C + P^2 f - P\, B(1)\, d^T f = 0 \end{aligned} \tag{70}$$

Solving (70) for C, we get $\dot{x}$ and x.

$$C^T = [-5.408307, 7.817756, -5.28571, 5.669461, -4.312463, 3.505344, -15.999902] \quad (71)$$

$$\tilde{x}(t) = [-0.01706, -0.049534, -0.096248, -0.125988, -0.1331, -0.116845, -0.071429]B(t)$$
. (72)

The analytic solution via Euler equation is

$$\ddot{x}(t) = -2t^2 + 4t - 2,$$
$$\dot{x}(t) = -\frac{2}{3}t^3 + 2t^2 - 2t,$$
$$x(t) = -\frac{1}{6}t^4 + \frac{2}{3}t^3 - t^2. \quad (73)$$

The least squares error E_2 is given as

$$E_2 = 4.98 \times 10^{-7} \quad (74)$$

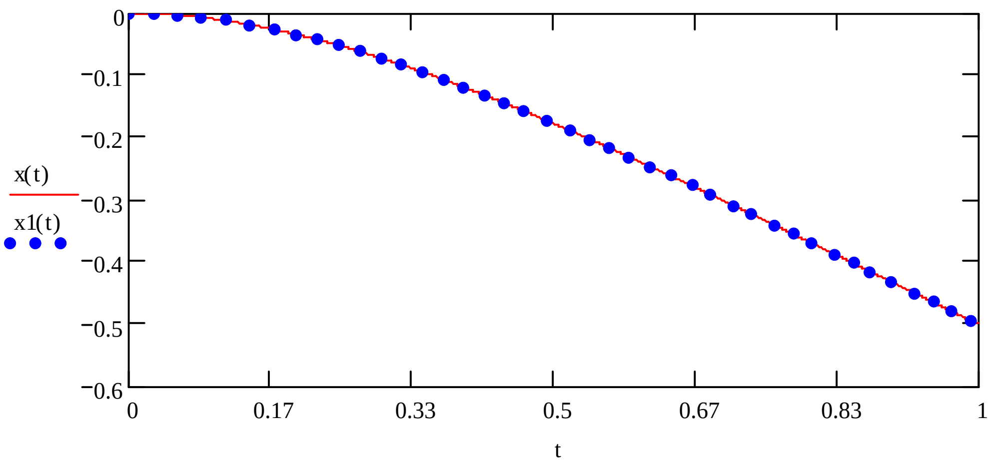

Fig.7. Compares the exact solution with the approximate solution for example 6.4.

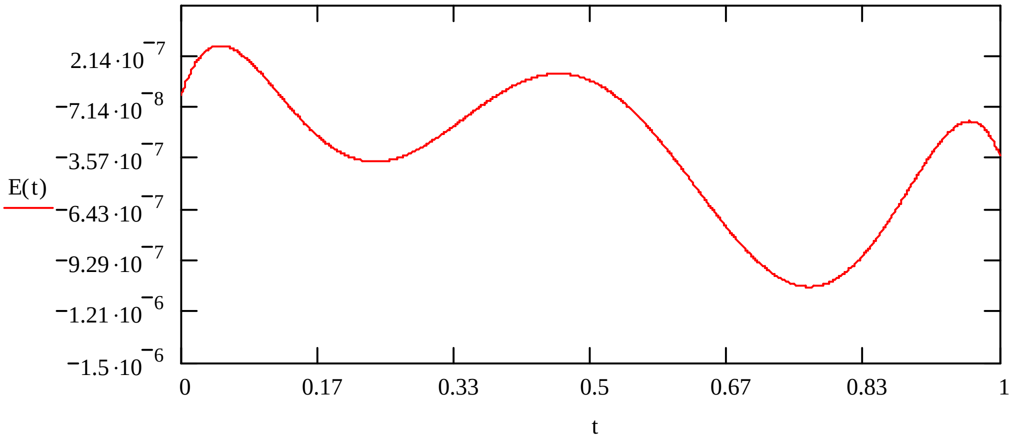

Fig.8. Shows the error between them for example 6

Table 1 (For Example 6.1)

t	Bernstein solution	Haar solution [7]	Analytical solution
0.125	0.058594	0.0586	0.058594
0.250	0.109375	0.1094	0.109375
0.375	0.152344	0.1523	0.152344
0.500	0.1875	0.1875	0.1875
0.625	0.214844	0.2148	0.214844
0.750	0.234375	0.2344	0.234375
0.875	0.246094	0.2461	0.246094
1	0.250000	0.250000	0.250000

Table 2 (For Example 6.2)

t	Bernstein solution	Haar solution [7]	Analytical solution
0.125	-0.003906	-0.0039	-0.003906
0.250	-0.015625	-0.0156	-0.015625
0.375	-0.035156	-0.0352	-0.035156
0.500	-0.062500	-0.0625	-0.062500
0.625	-0.097656	-0.0977	-0.097656
0.750	-0.140625	-0.1406	-0.140625
0.875	-0.191406	-0.1914	-0.191406
1	-0.250000	-0.2539	-0.250000

Table 3 (For Example 6.3)

t	Bernstein solution	RH functions [9]	Analytical solution
0.1	0.041951	0.0396	0.041951
0.2	0.079317	0.0761	0.079317
0.3	0.112473	0.1146	0.112473
0.4	0.141751	0.1482	0.141751
0.5	0.167443	0.1817	0.167443
0.6	0.189807	0.1817	0.189807
0.7	0.209066	0.2078	0.209066
0.8	0.225414	0.2267	0.225413
0.9	0.239013	0.2398	0.239013
1	0.250000	0.2515	0.250000

Table 4 (For Example 6.4)

t	Bernstein solution	Haar solution [7]	Analytical solution
0.125	-0.014364	-0.0138	-0.014364
0.250	-0.052735	-0.0518	-0.052734
0.375	-0.108765	-0.1075	-0.108765
0.500	-0.177083	-0.1758	-0.177083
0.625	-0.253296	-0.2521	-0.253296
0.750	-0.333985	-0.3330	-0.333984
0.875	-0.416708	-0.4161	-0.416707
1	-0.500000	-0.4999	-0.500000

6 Conclusions

The uniform approximation capabilities of Bernstein polynomials coupled with the fact that only a small number of polynomials (seven to be precise) are needed to obtain a satisfactory result makes our method very attractive. In [1], authors have used operational matrix of order 512 to achieve the accuracy comparable to ours as illustrated by Ex. 6.1 and 6.2. Tables 1-4 establish the superiority of our proposed method over the other existing methods, notably, the methods recently proposed by Hsiao [7], Babolian et. al. [1] and Razzaghi et. al. [9].

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