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Weak McCoy Rings Relative to a Monoid

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Abstract

Nielsen in [10] proves that reversible rings are McCoy and gives an example of a semi-commutative ring that is not right McCoy. At the same time, he also shows that semi-commutative rings do have a property close to the McCoy condition. For a monoid M, we introduce weak M-McCoy rings, which are a generalization of McCoy rings and M-Armendariz rings, and we investigate their properties. Every semi-commutative ring is weak M-McCoy for any unique product monoid and any strictly totally ordered monoid M. Moreover, we prove that for an ideal I of R, if I is semi-commutative and R/I is weak M-Armendariz, then R is weak M-McCoy for any strictly totally ordered monoid M. We show that for any nonzero ring R and any monoid M, the n-by-n upper triangular matrix ring $T_n(R)$ and the ring $\frac{R[x]}{\langle x^n \rangle}$, where $\langle x^n \rangle$ is the ideal generated by x^n and n is a positive integer, are weak M-McCoy. Finally we construct various examples of weak McCoy rings by reviewing and extending some results concerning to the structure of nilpotent elements of a ring R.

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1 Introduction

Throughout this paper R and M denote an associative ring with identity and a monoid respectively. For a ring R, we denote by $nil(R)$ the set of all nilpotent elements of R. Rege and Chhawchharia [12] introduced the notion of an Armendariz ring. By [12] a ring R is called an *Armendariz* ring if whenever polynomials $f(x) = a_0 + a_1x + ... + a_mx^m$, $g(x) = b_0 + b_1x + ... + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. The name Armendariz ring is chosen because Armendariz [2] had noted that a reduced ring (i.e., a ring without nonzero nilpotent elements) satisfies this condition.

Liu and Zhao [8] have studied a generalization of Armendariz rings, which they called weak-Armendariz rings. A ring R is called *weak-Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + ... + a_mx^m$, $g(x) = b_0 + b_1x + ... + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j \in ni\ell(R)$ for each i, j. According to Nielsen [10], a ring R is called a *right(resp. left) McCoy* ring if whenever $f(x)$, $g(x) \in R[x] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, then there exists a nonzero element $r \in R$ with $f(x)r = 0$ (resp. $rg(x) = 0$). A ring is *McCoy* if it is both left and right McCoy. By [4], a ring R is *right (resp. left) weak McCoy* if whenever $f(x) = a_0 + a_1x + ... + a_mx^m$, $g(x) = b_0 + b_1x + ... + b_nx^n \in R[x] \setminus \{0\}$ satisfies $f(x)g(x) = 0$, then $a_ic \in ni\ell(R)$ (resp. $cb_j \in ni\ell(R)$) for some nonzero $c \in R \setminus \{0\}$.

In [7], Liu studied a generalization of Armendariz rings, which are called *M-Armendariz rings* (Armendariz rings relative to the monoid M). A ring R is called *M-Armendariz*, if whenever $\alpha = a_1g_1 + a_2g_2 + ... + a_mg_m$, $\beta = b_1h_1 + b_2h_2 + ... + b_nh_n \in R[M]$ satisfy $\alpha\beta = 0$, then $a_ib_j = 0$ for each i, j. Also Y.S. Zhou and S.X. Mei [14], introduced and studied M-*McCoy* rings as a generalization of McCoy rings and M-Armendariz rings. A ring R is called *right(resp. left)* M-*McCoy* if for any $\alpha, \beta \in R[M] \setminus \{0\}$ with $\alpha\beta = 0$, there exists a nonzero $r \in R$ with $\alpha r = 0$ (resp. $r\beta = 0$). A ring R is M-*McCoy*, if it is left and right M-McCoy. A ring R is called *reversible* if $ab = 0$ implies $ba = 0$, for all $a, b \in R$; R is called *semi-commutative* if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. Reduced rings are clearly reversible and reversible rings are semi-commutative. In [10], Nielsen provides an example of a semi-commutative ring that is not right McCoy. At the same time, he shows that semi-commutative rings do have a property close to the McCoy condition.

In this paper we continue to study McCoy rings. We generalize and unify the above concepts by introducing the notion of weak M-McCoy rings. For a monoid M we introduce weak M-McCoy rings (weak-McCoy rings relative to a monoid M) which are a common generalization of weak M-Armendariz rings and McCoy rings. We do this by considering the weak-McCoy condition on monoid M instead of the ring R. This provides us with an opportunity to study McCoy rings in a general setting, and several known results on weak-McCoy rings are obtained as corollaries.

2 Weak McCoy rings relative to a monoid

Definition 2.1. *For a monoid M, a ring R is said to be right weak M-McCoy if whenever elements $\alpha = a_1g_1 + a_2g_2 + ... + a_mg_m$, $\beta = b_1h_1 + b_2h_2 + ... + b_nh_n \in R[M] \setminus \{0\}$ satisfy $\alpha\beta = 0$, then there exists a nonzero $c \in R$ with $a_ic \in ni\ell(R)$. We define left weak M-McCoy rings similarly. If a ring is both* *left and right weak M-McCoy, then we say that the ring is weak M-McCoy.*

Definition 2.2. *For a monoid M, a ring R is said to be weak M-Armendariz if whenever elements $\alpha = a_1g_1 + a_2g_2 + ... + a_mg_m$, $\beta = b_1h_1 + b_2h_2 + ... + b_nh_n \in R[M]$ satisfy $\alpha\beta = 0$, then $a_ib_j \in ni\ell(R)$ for each i, j.*

Let $M = (\mathbb{N} \cup \{0\}, +)$. Then a ring R is weak M-McCoy (resp. weak M-Armendariz) if and only if R is weak McCoy(resp. weak Armendariz). If $M = \{e\}$, then for any ring R, R is weak M-McCoy and weak M-Armendariz.

Theorem 2.3. *For any monoid M;*
(1) weak M-Armendariz rings are weak M-McCoy.
(2) M-McCoy rings are weak M-McCoy.
(3) M-Armendariz rings are weak M-McCoy.

Proof. (1) Let R be a weak M-Armendariz ring and $\alpha = a_1g_1+a_2g_2+...+a_mg_m$, $\beta = b_1h_1 + b_2h_2 + ... + b_nh_n \in R[M] \setminus \{0\}$ satisfy $\alpha\beta = 0$. Then $a_ib_j \in ni\ell(R)$ for each i, j. Since $\alpha \neq 0$ and $\beta \neq 0$, there exist $1 \leq k \leq m$, $1 \leq l \leq n$ such that a_k, $b_l \in R \setminus \{0\}$. Hence $a_kb_j \in ni\ell(R)$ and $a_ib_l \in ni\ell(R)$. Therefore R is weak M-McCoy.
(2) It follows easily from the definition.
(3) It is easy to see that each M-Armendariz ring is M-McCoy and therefore weak M-McCoy by (2).

Recall that a monoid M is called a *u.p.-monoid*(unique product monoid) if for any two nonempty finite subsets $A, B \subseteq M$ there exist an element $g \in M$ uniquely presented in the form ab where $a \in A$ and $b \in B$. The class of u.p.-monoids is quite large and important(see [3] , [11]). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid M has no non-unity element of finite order.

Proposition 2.4. *Let M be a u.p.-monoid and R a reversible ring. Then R is a M-McCoy ring.*

Proof. We only prove the right case, the proof of the other case is similar. Assume on the contrary that $\alpha c_1 \neq 0$ if $c_1 \in R$ is nonzero and say that β is a nonzero element with the smallest number of terms in $R[M]$ with respect to the property $\alpha\beta = 0$. Let $\alpha = a_1g_1 + a_2g_2 + ... + a_mg_m$, $\beta = b_1h_1 + b_2h_2 + ... + b_nh_n \in R[M] \setminus \{0\}$. Then $n \geq 2$ and $\alpha b_n \neq 0$ by the assumption; hence $a_sb_n \neq 0$ for some s and thus $a_s\beta \neq 0$. Let t be the largest integer such that $a_t\beta \neq 0$, then we have

$\alpha\beta = (a_{i_1}g_{i_1} + a_{i_2}g_{i_2} + ... + a_{i_k}g_{i_k})(b_1h_1 + b_2h_2 + ... + b_nh_n) = 0$, where $i_k = t$, $\{i_1, i_2, ..., i_k\} \subseteq \{1, 2, ..., t\}$, and $a_{i_h}\beta \neq 0$ for all $h \in \{1, 2, ..., k\}$. Now let $A = \{g_{i_1}, g_{i_2}, ..., g_{i_k}\}$, $B = \{h_1, h_2, ..., h_n\}$ and $C = \{g_ih_j | i_1 \leq i \leq i_k, 1 \leq j \leq n\}$. Since M is a u.p.-monoid, we can say that $g_{i_k}h_n = g_th_n$ has a unique representation in C, after reordering if necessary, whence $a_tb_n = 0$. So $b_na_t = 0$, since R is reversible. Consequently we have $0 \neq a_t\beta = a_t(b_1h_1 + b_2h_2 + ... + b_{n-1}h_{n-1})$, and hence $\beta a_t \neq 0$, since R is reversible. So $\alpha(\beta a_t) = (\alpha\beta)a_t = 0$. This is a contradiction because nonzero βa_t has smaller number of terms than β. So there exist a nonzero c in R such that $\alpha c = 0$. Therefore R is right M-McCoy.

Let $(M, \leq)$ be an ordered monoid. If for any $g_1, g_2, h \in M$, $g_1 < g_2$ implies that $g_1h < g_2h$ and $hg_1 < hg_2$, then $(M, \leq)$ is called a *strictly ordered monoid.*

Corollary 2.5. *Let M be a strictly totally ordered monoid and R be a reversible ring. Then R is M-McCoy.*

For any $\alpha \in R[M]$, we denote by C_α the set of all coefficients of α.

Proposition 2.6. *If M is a u.p.-monoid, then every semi-commutative ring is weak M-McCoy.*

Proof. Suppose $\alpha = a_1g_1 + a_2g_2 + \cdots + a_mg_m$, $\beta = b_1h_1 + b_2h_2 + \cdots + b_nh_n \in R[M]\backslash\{0\}$ satisfy $\alpha\beta = 0$. Since R is semi-commutative, $ni\ell(R)$ is an ideal of R by [8, Lemma 3.1]. Clearly, the ring $\frac{R}{ni\ell(R)}$ is reduced. Thus $\frac{R}{ni\ell(R)}$ is M-Armendariz by [7, Proposition 1.1]. For any $\alpha = \sum_{i=1}^{m} a_ig_i \in R[M]$, denote $\bar{\alpha} = \sum_{i=1}^{m} (a_i + ni\ell(R))g_i \in \frac{R}{ni\ell(R)}[M]$. It is easy to see that the mapping $\psi : R[M] \to \frac{R}{ni\ell(R)}[M]$ defined by $\psi(\alpha_i) = \bar{\alpha}_i$ is a ring homomorphism. Since $\alpha\beta = 0$, $C_{\alpha\beta} \subseteq ni\ell(R)$. Hence we have $\bar{\alpha}\bar{\beta} = \overline{\alpha\beta} = 0$. So by [7, Proposition 1.6] $\bar{a}_i\bar{b}_j = 0$ for each i, j, since $\frac{R}{ni\ell(R)}$ is M-Armendariz. Thus $a_ib_j \in ni\ell(R)$ for each i, j. Choosing $c_1 = a_m \neq 0$, $c_2 = b_n \neq 0$, we have $c_1b_j \in ni\ell(R)$, $a_ic_2 \in ni\ell(R)$ for each i, j. Therefore R is weak M-McCoy.

Corollary 2.7. *If M is a strictly totally ordered monoid, then every semi-commutative ring is weak M-McCoy.*

Taking $M = (\mathbb{N} \cup \{0\}, +)$, we have the following corollary.

Corollary 2.8. *Each semi-commutative ring is weak McCoy.*

The following example shows that the condition "M is a u.p.-monoid" in

Proposition 2.6 is not superfluous.

Example 2.9. *Let M be a cyclic group of order $n \geq 2$, then the field of complex numbers $\mathbb{C}$ is not weak M-McCoy.*

Proof. Suppose that $M = \{e, g, g^2, \cdots, g^{n-1}\}$. Let $\alpha = 1e + 1g + 1g^2 + \cdots + 1g^{n-1}$ and $\beta = 1e + (-1)g$. Then $\alpha\beta = 0$. Thus $\mathbb{C}$ is not weak M-McCoy, whereas $\mathbb{C}$ is semi-commutative.

The ring of Laurent polynomials in x with coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} r_i x^i$ with obvious addition and multiplication, where $r_i \in R$ and k, n are (possibly negative) integers; denoted by $R[x, x^{-1}]$.

Corollary 2.10. *Let R be a semi-commutative ring. Then R is weak $\mathbb{Z}$-McCoy, that is, for any $\alpha = a_{-m}x^{-m} + a_{-(m-1)}x^{-(m-1)} + ... + a_p x^p$, $\beta = b_{-n}x^{-n} + b_{-(n-1)}x^{-(n-1)} + ... + b_q x^q \in R[x, x^{-1}] \setminus \{0\}$, if $\alpha\beta = 0$, then there exist $c_1, c_2 \in R \setminus \{0\}$ such that $a_i c_1 \in ni\ell(R)$ and $c_2 b_j \in ni\ell(R)$ for each $-m \leq i \leq p$ and $-n \leq j \leq q$.*

Proof. Note that $R[\mathbb{Z}] \cong R[x, x^{-1}]$.

It was shown in [7, Proposition 1.4], that if M is strictly totally ordered monoid and I is a reduced ideal of R such that R/I is an M-Armendariz ring, then R is M-Armendariz. Here we have the following result, which is a generalization of this.

Proposition 2.11. *Let M be a strictly totally ordered monoid and I an ideal of R. If I is semi-commutative and R/I is weak M-Armendariz, then R is weak M-McCoy.*

Proof. Let $\alpha = a_1g_1 + a_2g_2 + ... + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + ... + b_mh_m \in R[M] \setminus \{0\}$ be such that $\alpha\beta = 0$ and $g_1 < g_2 < \cdots < g_n$, $h_1 < h_2 < \cdots < h_m$. Thus we have the following equations:
$\sum_{g_i h_j = \omega} a_i b_j = 0$ (sum of all couples $(g_i, h_j) \in M \times M$ such that $g_i h_j = \omega$, for all $\omega \in M$).
Without loss of generality we can suppose that $a_1 \neq 0$, $b_1 \neq 0$, and it is suffices to show that $a_1 b_j \in ni\ell(R)$, $a_i b_1 \in ni\ell(R)$, for each i, j, using transfinite induction on the strictly totally ordered set $(M, \leq)$. Note that in $(R/I)[M]$, $(\bar{a_1}g_1 + \bar{a_2}g_2 + ... + \bar{a_m}g_m)(\bar{b_1}h_1 + \bar{b_2}h_2 + ... + \bar{b_n}h_n) = 0$. Thus there exists $n_{ij} \in \mathbb{N}$ such that $(a_i b_j)^{n_{ij}} \in I$, since R/I is weak M-Armendariz. Clearly, $g_1 h_1 < g_i h_j$ if $i \neq 1$ or $j \neq 1$. Hence $a_1 b_1 \in ni\ell(R)$. Now suppose that

$\omega \in M$ is such that for any h_j with $g_1 h_j < \omega$, $a_1 b_j \in ni\ell(R)$. We will show that $a_1 b_j \in ni\ell(R)$ for any h_j with $g_1 h_j = \omega$. Set $X = \{(g_1, h_j) | g_1 h_j = \omega\}$. We write X as $\{(g_{i_t}, h_{j_t}) | t = 1, 2, \cdots, k\}$ such that $g_{i_1} = g_{i_2} = \cdots = g_{i_k} = g_1$. Since M is cancellative, $g_{i_{t_1}} h_{j_{t_1}} = g_{i_{t_2}} h_{j_{t_2}} = \omega$ implies that $h_{j_{t_1}} = h_{j_{t_2}} = h_k$. So we must prove that $a_1 b_k \in ni\ell(R)$, where $a_1 b_{k-1} \in ni\ell(R)$ by the hypothesis. Now using semi-commutativity of I and similar to that of [8, Proposition 3.6], we can easily show that $a_1 b_k \in ni\ell(R)$. Therefore R is weak M-McCoy.

Remark 2.12. *Any weak McCoy ring is weak M-McCoy for some monoid M. But weak M-McCoy ring need not be a weak McCoy, because every ring is weak $\{e\}$-McCoy but not weak McCoy.*

Recall that a monoid M is called *torsion-free* if the following property holds:
if $g, h \in M$ and $k \geq 1$ are such that $g^k = h^k$, then $g = h$.

Corollary 2.13. *Let M be a commutative, cancellative and torsion-free monoid. If one of the following conditions holds, then R is weak M-McCoy:*
(1) R is a semi-commutative ring.
(2) R/I is a weak M-Armendariz ring for some ideal I of R, and I is semi-commutative.

Proof. If M is commutative, cancellative, and torsion-free, then, by [13] there exists a compatible strict total order $\leq$ on M. Now the results follows from Corollary 2.7 and Proposition 2.11.

Proposition 2.14. *Let M be a monoid and R be any nonzero ring. Then the upper triangular matrix ring $T_n(R)$ is weak M-McCoy.*

Proof. Let $\alpha = a_1 g_1 + a_2 g_2 + ... + a_m g_m$, $\beta = b_1 h_1 + b_2 h_2 + ... + b_n h_n \in T_n(R)[M] \setminus \{0\}$ satisfy $\alpha\beta = 0$. If we take $0 \neq c = \sum_{i=1}^{n-1} E_{i,i+1} \in T_n(R)$ then we have $a_i c \in ni\ell(R)$ and $c b_j \in ni\ell(R)$ for all i, j. Therefore $T_n(R)$ is weak M-McCoy.

Let R be a ring and consider the following subring of the triangular ring $T_n(R)$:

$$T(R,n) := \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_i \in R \right\},$$

with $n \geq 2$. Then $T(R, n)$ is a ring with addition point-wise and usual matrix multiplication.

Corollary 2.15. *Let M be a monoid and R be any ring. Then the ring $T(R, n)$ is weak M-McCoy.*

Corollary 2.16. *Let M be a monoid and R be any ring. Then the ring $\frac{R[x]}{\langle x^n \rangle}$ is weak M-McCoy.*

Proof. Observe that $T(R, n) \cong \frac{R[x]}{\langle x^n \rangle}$, for each positive integer n.

Corollary 2.17. *Let M be a monoid and R be any ring. Then the trivial extension of R, $T(R, R) := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$, is a weak M-McCoy ring.*

The following example show that the converse of Theorem 2.3 is not true.

Example 2.18. (a) Let R be any ring and $M = (\mathbb{N} \cup \{0\}, +)$. Then by [8, Example 2.5], $S = M_2(R)$ is not weak M-Armendariz. Therefore $T_n(S)$ is not weak M-Armendariz by [8, Theorem 2.2], whereas $T_n(S)$ is weak M-McCoy by Proposition 2.14.
(b) Let M be a monoid with $|M| \geq 2$ and R ba any ring. Then by [14, Example 2.6], $T_n(R)$ is not M-McCoy, whereas $T_n(R)$ is weak M-McCoy by Proposition 2.14.
(c) Let M be a monoid with $|M| \geq 2$ and R ba any ring. Take $e \neq g \in M$. Let $\alpha = (E_{11} + E_{1n})e + (E_{11} - E_{1n})g$, $\beta = (-E_{1n} + \sum_{i=2}^{n} E_{ii})e + (E_{1n} + E_{nn})g$ $\in T_n(R)[M] \setminus \{0\}$, where $E_{i,j}$ are the matrix units in $T_n(R)$. Then $\alpha\beta = 0$, but $(E_{11} + E_{1n})(E_{1n} + E_{nn}) \neq 0$. Thus $T_n(R)$ is not M-Armendariz ring, whereas $T_n(R)$ is weak M-McCoy by Proposition 2.14.

Proposition 2.19. *Let M be a cancellative monoid and N an ideal of M. If R is weak N-McCoy, then R is weak M-McCoy.*

Proof. Suppose that $\alpha = a_1g_1 + a_2g_2 + ... + a_ng_n$, $\beta = b_1h_1 + b_2h_2 + ... + b_mh_m$ $\in R[M] \setminus \{0\}$ such that $\alpha\beta = 0$. Take $g \in N$. Then gg_1, gg_2, ..., gg_n, h_1g, h_2g, ..., $h_mg \in N$ and $gg_i \neq gg_j$ and $h_ig \neq h_jg$ when $i \neq j$. Now from $(\sum_{i=1}^{n} a_igg_i)(\sum_{j=1}^{m} b_jh_jg) = 0$ and from the hypothesis that R is weak N-McCoy it follows that there exists $c \in R \setminus \{0\}$ such that $a_ic \in ni\ell(R)$ for each i. Thus R is weak M-McCoy.

Lemma 2.20. *Let M be a monoid and N a submonoid of M. If R is weak M-McCoy, then R is weak N-McCoy.*

Lemma 2.21([7], Lemma 1.13). *Let M and N be a u.p.-monoids. Then so is the monoid $M \times N$.*

Let $T(G)$ be the set of elements of finite order in an Abelian group G. Then $T(G)$ is a fully invariant subgroup of G. G is said to be torsion-free, if $T(G) = \{e\}$.

Theorem 2.22. *Let G be a finitely generated abelian group. Then the following conditions on G are equivalent:*
(1) G is torsion-free.
(2) There exists a ring R with $|R| \geq 2$ such that R is weak G-McCoy.

Proof. (1) $\Rightarrow$ (2) Since G is a finitely generated torsion-free abelian group, then $G \cong \mathbb{Z} \times \mathbb{Z} \times ... \times \mathbb{Z}$, a finite direct product of $\mathbb{Z}$. By Lemma 2.21, G is a u.p.-monoid. Let R be a semi-commutative ring. Then R is weak G-McCoy, by Proposition 2.6.
(2) $\Rightarrow$ (1) If $g \in T(G)$ and $g \neq e$, then $N =< g >$ is a cyclic group of finite order. If a ring $R \neq \{0\}$ is weak G-McCoy, then, by Lemma 2.20, R is weak N-McCoy, which contradicts Example 2.9. Thus G is torsion-free.

3 Extensions of weak McCoy rings

Recall that a ring R is called right Ore if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is well-known that R is a right Ore ring if and only if the classical right quotient ring of R exists.

Theorem 3.1. *Let R be a right Ore ring with its classical right quotient ring Q. If R is weak McCoy ring, then Q is weak McCoy .*

Proof. Let $F(x) = \sum_{i=0}^{m} a_i u^{-1} x^i$ and $G(x) = \sum_{j=0}^{n} b_j v^{-1} x^j \in Q[x] \setminus \{0\}$ satisfy $F(x)G(x) = 0$, where $a_i, b_j \in R$ and u, v are regular elements of R. Since R is right Ore ring, there exists $b_j \in R$ and regular element u_1 such that $u^{-1} b_j = b_j u_1^{-1}$ for $j = 0, 1, 2, ..., n$. Then $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. Since R is weak McCoy, there exists $0 \neq c \in R$ such that $a_i c \in ni\ell(R)$. So $a_i u^{-1} c_1 \in ni\ell(Q)$ where $c_1 = uc$. Therefore Q is weak McCoy.

Proposition 3.2. *Let R be a ring and Δ be a multiplicative closed subset of R consisting of central regular elements. Then R is weak McCoy if and only* *if* $\Delta^{-1}R$ *is weak McCoy.*

Proof. We only prove "if" portion, because the "only if" portion is proved by [4, proposition 3.3]. Suppose $\Delta^{-1}R$ be a weak McCoy ring and $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x] \setminus \{0\}$ satisfies $f(x)g(x) = 0$. Since $\Delta^{-1}R$ is weak McCoy, $a_i c\alpha^{-1} \in ni\ell(\Delta^{-1}R)$ for some nonzero $c\alpha^{-1} \in \Delta^{-1}R$. Thus $a_i c \in ni\ell(R)$. Therefore R is weak McCoy ring.

Corollary 3.3. *Let* R *be a ring. Then* $R[x]$ *is weak McCoy if and only if* $R[x, x^{-1}]$ *is weak McCoy.*

Proof. Clearly $\Delta = \{1, x, x^2, ...\}$ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements and $\Delta^{-1}R[x] = R[x, x^{-1}]$. Hence the proof follows from Proposition 3.2.

Proposition 3.4. *Let* R *be a semi-commutative ring. Then* $ni\ell(R[x]) = ni\ell(R)[x]$.

Proof. First we show that $ni\ell(R[x]) \subseteq ni\ell(R)[x]$. Suppose that $f(x) = \sum_{i=0}^{m} a_i x^i \in ni\ell(R[x])$. Then $[f(x)]^l = 0$, for some natural number l. Hence $a_0 \in ni\ell(R)$. Since R is a semi-commutative ring, $ni\ell(R)$ is an ideal of R, by [8, Lemma 3.1] and hence $ni\ell(R)[x]$ is an ideal of $R[x]$. Therefore $a_1^l = 0$ and hence $a_1 \in ni\ell(R)$. Inductively one can see that $a_i \in ni\ell(R)$, for $0 \leq i \leq m$. Hence $f(x) \in ni\ell(R)[x]$. Now we shall prove that $ni\ell(R)[x] \subseteq ni\ell(R[x])$. For this mean we have prove that if $a_i \in ni\ell(R)$, for $0 \leq i \leq n$, then $f(x) = \sum_{i=0}^{n} a_i x^i$ is a nilpotent element of $R[x]$. Suppose that $a_i^m = 0$, for some natural number m. Let $k = (n+1)m+1$. We have $[f(x)]^k = \sum_{s=0}^{nk} (\sum_{i_1+...i_k=s} a_{i_1} \dots a_{i_k}) x^k$. So the number of a_t in $a_{i_1} \dots a_{i_k}$ is more than m, for some $0 \leq t \leq n$. Since R is semi-commutative and $a_t^n = 0$, we have $a_{i_1} \dots a_{i_k} = 0$. Therefore $[f(x)]^k = 0$ and we we prove that $ni\ell(R)[x] \subseteq ni\ell(R[x])$.

Theorem 3.5. *Let* R *be a semi-commutative ring. Then the following statements are equivalent:*
(1) R *is a weak McCoy ring.*
(2) $R[x]$ *is a weak McCoy ring.*
(3) $R[x, x^{-1}]$ *is a weak McCoy ring.*

Proof. (1) $\Leftrightarrow$ (2): Let R be a weak McCoy ring and $F(y) = \sum_{i=0}^{m} f_i y^i$ and $G(y) = \sum_{j=0}^{n} g_j y^j \in R[x][y] \setminus \{0\}$ such that $F(y)G(y) = 0$, where $f_i = \sum_{s=0}^{p_i} a_{is} x^s$ and $g_j = \sum_{t=0}^{q_j} b_{jt} x^t \in R[x]$. As in the proof of [1], there exists u and $v \in R \setminus \{0\}$ such that ub_{jt} and $a_{is}v \in ni\ell(R)$. Thus ug_j and $f_i v \in ni\ell(R[x])$, by Proposition 3.4. So $R[x]$ is weak McCoy ring. Conversely,

suppose that $R[x]$ is weak McCoy and $f(y)g(y) = 0$, where $f(y) = \sum_{i=0}^{m} a_i y^i$ and $g(y) = \sum_{j=0}^{n} b_j y^j \in R[y] \subseteq R[x][y]$. Since $R[x]$ is weak McCoy ring, there exist $h(x) = \sum_{k=0}^{p} h_k x^k$ and $i(x) = \sum_{l=0}^{q} i_l x^l \in R[x]\backslash\{0\}$, such that $a_i h(x)$ and $i(x)b_j \in ni\ell(R[x]) = ni\ell(R)[x]$, for each i, j. So $a_i h_k$ and $i_l b_j \in ni\ell(R)$, for each k, l. Let h_c and $i_{c'}$ be the non-zero coefficient in $h(x)$ and $i(x)$, respectively. So $a_i h_c$ and $i_{c'} b_j \in ni\ell(R)$. Therefore R is a weak McCoy ring.
(2) $\Leftrightarrow$ (3) By Corollary 3.3.

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