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On α-Skew Quasi Armendariz Modules

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Abstract

Let R be a ring and α be a ring endomorphism of R. We introduce α-skew quasi-Armendariz modules as a generalization of quasi-Armendariz rings and modules. Some properties of this generalization and the relationship between an R-module M_R and the general polynomial module $M[x]$ over the skew polynomial ring $R[x;\alpha]$ are established. Among other results, we show that there is a strong connection of the Baer, quasi-Baer and the p.p.-property of the two modules, respectively. As a consequence we extend and unify several known results.

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1 Introduction

Throughout this paper R denotes an associative ring with unity, and α is a ring endomorphism of R. We denote $R[x;\alpha]$ the skew polynomial ring whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x$ for any $a \in R$. A ring R is called *Baer* (respectively *quasi-Baer*) if the right annihilator of every nonempty subset (respectively right ideal) of R is generated, as a right ideal, by an idempotent of R. Kaplansky [15], introduced the Baer rings to abstract various properties of rings of operators on a Hilbert space. Clark [11] introduced the quasi-Baer rings and used them to characterize a finite dimensional twisted matrix units semigroup algebra over an algebraically closed field. All modules are assumed to be unitary right modules. For a subset X of a module M_R, let $ann_R(X) = \{r \in R | Xr = 0\}$. In [16], Lee and Zhou introduced Baer, quasi-Baer and p.p.-modules as follows:

(1) M_R is called *Baer* (respectively *quasi-Baer*) if, for any subset (respectively submodule) X of M, $ann_R(X) = eR$, where $e^2 = e \in R$. (2) M_R is called *principally projective* (or simply p.p.) module (respectively *right principally quasi-Baer* (or simply *right p.q.-Baer*) module) if, for any element $m \in M$, $ann_R(m) = eR$ (resp. $ann_R(mR) = eR$) where $e^2 = e \in R$.

Clearly, a ring R is Baer (respectively p.p. or quasi-Baer) if and only if R_R is Baer (respectively p.p. or quasi-Baer) module. If R is a Baer (respectively p.p. or quasi-Baer) ring, then for any right ideal I of R, I_R is Baer (respectively p.p. or quasi-Baer) module. It is clear that R is a right p.q.-Baer ring if and only if R_R is a right p.q.-Baer module. Every submodule of a right p.q.-Baer module is right p.q.-Baer and every Baer module is quasi-Baer.

A ring R is called *reduced* if it has no nonzero nilpotent element and M_R is called reduced by Lee and Zhou [16] if, for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. Lee and Zhou have extended various results of reduced rings to reduced modules.

Hirano in [13], introduced and studied the notion of quasi-Armendariz rings. A ring R is called *quasi-Armendariz* if whenever two polynomials $f(x) = \sum_{i=0}^{k} a_i x^i$, $g(x) = \sum_{j=0}^{\ell} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, it implies that $a_i R b_j = 0$ for all i, j. Every semiprime ring is a quasi-Armendariz ring, and the class of quasi-Armendariz rings is Morita stable. Moreover, if R is a quasi-Armendariz ring, then, for any positive integer n and for any set X of commutative indeterminates, the upper triangular matrix ring $T_n(R)$ and the polynomial ring $R[X]$ are also quasi-Armendariz.

By [5], a module M_R is called quasi-Armendariz if whenever two polynomials $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x]$ and $g(x) = \sum_{j=0}^{\ell} b_j x^j \in R[x]$ satisfy $m(x)R[x]g(x) = 0$, it implies that $m_i R b_j = 0$ for all i, j.

Zhang and Chen [18] introduced the notion of *α-skew Armendariz modules.* Namely, an R-module M_R is called α-skew Armendariz, if for polynomials $m(x) = m_0 + m_1 x + \ldots + m_k x^k \in M[x]$ and $f(x) = b_0 + b_1 x + \ldots + b_n x^n \in R[x;\alpha]$, $m(x) f(x) = 0$ implies $m_i \alpha^i(b_j) = 0$ for each $0 \le i \le k$ and $0 \le j \le n$. According to Lee and Zhou [16] a module M_R is called *α-Armendariz* if M_R is α-compatible and α-skew-Armendariz. If α is equal to identity, then the above definition boils down to the standard notion of Armendariz module. By [16] a module M_R is α-reduced if M is α-compatible and reduced.

In this paper, for a ring endomorphism $\alpha : R \to R$, we introduce α-skew quasi Armendariz modules which are a generalization of quasi Armendariz rings and quasi Armendariz modules, and investigate their properties. Moreover, we study on the relationship between the Baerness and p.p.-property of a module M_R and these of the general polynomial module $M[x]$ over the skew polynomial ring $R[x;\alpha]$. We study the relationship between the set of annihilators in M and the set of annihilators in $M[x;\alpha]_{R[x;\alpha]}$. We give a sufficient condition for a module to be α-skew quasi-Armendariz. We show that some extensions of a α-skew quasi-Armendariz module are α-skew quasi-Armendariz. Furthermore, a necessary and sufficient condition for the trivial extension $T(R,M)$ to be α-skew quasi-Armendariz is obtained. This work extends and unifies several known results related to quasi-Armendariz rings and modules.

2 On α-skew Armendariz modules

Following Lee and Zhou [16], $M[x;\alpha] = \{\sum_{i=0}^{s} m_i x^i \ : s \geq 0, m_i \in M\}$ is an abelian group under an obvious addition operation. Moreover, $M[x;\alpha]$ becomes a module over $R[x;\alpha]$ under the following scalar product operation: for $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{t} b_j x^j \in R[x;\alpha]$, $m(x).f(x) = \sum_{k=0}^{s+t}(\sum_{i+j=k} m_i \alpha^i(b_j))x^k$.

Definition 2.1. (*Annin,* [3]) *Let R be a ring with an endomorphism α. An R-module M_R is α-compatible if for each $m \in M$ and $r \in R$, we have $mr = 0 \Leftrightarrow m\alpha(r) = 0$.*

The α-compatibility condition on M_R is a natural, independently interesting condition from which we can derive a number of interesting properties, and it will be of invaluable service in the proof of our main results.

Definition 2.2. (*Zhang and Chen,* [18]) *Let R be a ring with an endomorphism α. A module M_R is said to be α-skew Armendariz, if for polynomials $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{t} b_j x^j \in R[x;\alpha]$, $m(x).f(x) = 0$ implies $m_i\alpha^i(b_j) = 0$ for each $0 \leq i \leq s$ and $0 \leq j \leq t$.*

For a ring R and R-module M_R, put $rAnn_R(2^{M_R}) = \{r_R(U) \mid U \subseteq M_R\}$.

Theorem 2.3. *Let M_R be an α-compatible module. Then the following statements are equivalent:*
(1) M_R is α-skew Armendariz.
(2) $\phi : rAnn_R\left(2^M\right) \to rAnn_{R[x;\alpha]}\left(2^{M[x;\alpha]}\right)$ defined by
$\phi\left(r_R(U)\right) = r_{R[x;\alpha]}(U) = r_R(U)[x;\alpha]$,
for every $r_R(U) \in rAnn_R\left(2^M\right)$, is bijective.

Proof. $(2) \to (1)$. Let $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{t} b_j x^j \in R[x;\alpha]$ and $0 = m(x).f(x) = \sum_{k=0}^{s+t}\left(\sum_{i+j=k} m_i \alpha^i(b_j)\right)x^k$. Therefore $f(x) \in r_{R[x;\alpha]}(m(x))$ and hence by hypothesis $f(x) \in r_{R[x;\alpha]}(m(x)) = r_R(U)[x;\alpha]$ for some $U \subseteq M$. So $b_j \in r_R(U)$ for all j. Thus $b_j \in r_R(U) \subseteq r_R(U)[x;\alpha] =$

$r_{R[x;\alpha]}(m(x))$ so $m(x)b_j = 0$. Consequently $m_i\alpha^i(b_j) = 0$ for all i, j. Therefore M_R is α-skew Armendariz.

(1) $\rightarrow$ (2) Obviously ϕ is injective, so it is enough to show that ϕ is surjective. Let $r_{R[x;\alpha]}(V) \in rAnn_{R[x;\alpha]}\left(2^{M[x;\alpha]}\right)$ for some $V \subseteq M[x;\alpha]$. Then for $r_R(c_V) \in rAnn_R\left(2^M\right)$, $\phi(r_R(c_V)) = r_{R[x;\alpha]}(c_V) = r_{R[x;\alpha]}(V)$. Let $f(x) \in r_{R[x;\alpha]}(c_V)$ where $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x;\alpha]$. So $c_V.f(x) = 0$ and for all $m \in c_V$, $mf(x) = ma_0 + ma_1x + \cdots + ma_nx^n = 0$ and hence $ma_j = 0$ for all j. Now, for each $p(x) = \sum_{j=0}^{t} p_j x^j \in V$ we get $p(x)f(x) = 0$, since $p_j \in c_V$ for all j. Hence $f(x) \in r_{R[x;\alpha]}(V)$.
Conversely, let $g(x) = \sum_{j=0}^{k} b_j x^j \in r_{R[x;\alpha]}(V)$. Then for all $m(x) = \sum_{j=0}^{\ell} m_j x^j \in V$, $m(x)g(x) = 0$. Since M_R is α-skew Armendariz, $m_i\alpha^i(b_j) = 0$ for all i, j. Since M_R is α-compatible, we have $m_ib_j = 0$ for all i, j. Hence $m_ig(x) = 0$ for all i. So $g(x) \in r_{R[x;\alpha]}(c_V)$, since $m(x) \in V$ is arbitrary. Therefore ϕ is surjective.

Corollary 2.4. *Let M_R be an α-compatible module. Then the following statements are equivalent:*
(1) M_R is α-Armendariz.
(2) $\phi : rAnn_R\left(2^M\right) \rightarrow rAnn_{R[x;\alpha]}\left(2^{M[x;\alpha]}\right)$ defined by
$\phi(r_R(U)) = r_{R[x;\alpha]}(U) = r_R(U)[x;\alpha]$ for every $r_R(U) \in rAnn_R\left(2^M\right)$, is bijective.

Corollary 2.5. *Let M_R be an α-Armendariz module. Then M_R is a Baer module if and only if $M[x;\alpha]_{R[x;\alpha]}$ is a Baer module.*

Corollary 2.6. ([14], Theorem 3.2) *Let R be an α-Armendariz ring. Then R is a Baer ring if and only if $R[x;\alpha]$ is a Baer ring.*

Corollary 2.7. *Let M_R be an α-Armendariz module. Then M_R is a p.p.-module if and only if $M[x;\alpha]_{R[x;\alpha]}$ is a p.p.-module.*

Corollary 2.8. ([14], Theorem 3.3) *Let R be an α-Armendariz ring. Then R is a p.p.-ring if and only if $R[x;\alpha]$ is a p.p.-ring.*

Theorem 2.9. *Let M_R be an α-reduced module. Then M_R is p.q.-Baer if and only if $M[x;\alpha]_{R[x;\alpha]}$ is p.q.-Baer.*

Proof. Assume that M_R is a p.q.-Baer module. Since M_R is α-reduced, by [16, Lemma 1.2], $ma = 0$ implies $mRa = 0$ for any $m \in M$ and $a \in R$. Hence $r_R(m) = r_R(mR)$ and so M_R is a p.p.-module. On the other hand M_R is α-Armendariz and so by [16, Theorem 2.11 (1)(a)], $M[x;\alpha]_{R[x;\alpha]}$ is a

p.p.-module. Hence $M[x;\alpha]_{R[x;\alpha]}$ is p.q.-Baer, because by [16, Theorem 1.6] it is reduced.
Conversely, suppose that $M[x;\alpha]_{R[x;\alpha]}$ is a p.q.-Baer module. So for each $m \in M$, $r_{R[x;\alpha]}(mR[x;\alpha]) = f(x)R[x;\alpha]$, where $f(x)^2 = f(x) \in R[x;\alpha]$. We have $f(x)R[x;\alpha] \subseteq r_{R[x;\alpha]}(mR) = r_R(mR)[x;\alpha]$. Let $g(x) = \sum_{j=0}^{n} b_j x^j \in r_R(mR)[x;\alpha]$. Then $mRb_j = 0$ and so $mR\alpha^k(b_j) = 0$ for all j, since M_R is α-compatible. Let $k(x) = \sum_{i=0}^{m} k_i x^i \in mR[x;\alpha]$. Then we have $k(x)g(x) = \sum_{k=0}^{m+n}(\sum_{i+j=k} k_i\alpha^i(b_j))x^k = 0$, hence $g(x) \in r_{R[x;\alpha]}(mR[x;\alpha])$ and therefore $r_R(mR)[x;\alpha] = r_{R[x;\alpha]}(mR[x;\alpha]) = f(x)R[x;\alpha]$. Let $f(x) = \sum_{i=0}^{n} a_i x^i$, where all $a_i \in r_R(mR)$. For any $a \in r_R(mR)$, there exists $h(x) \in R[x;\alpha]$ such that $a = f(x)h(x)$. Hence $f(x).a = f(x)f(x)h(x) = f(x)h(x) = a$. This implies that $a_0a = a$. Since $f(x)^2 = f(x)$ and hence $a_0{}^2 = a_0$, and $r_R(mR) = a_0R$, M_R is a p.q.-Baer module.

3 On α-skew quasi-Armendariz modules

According to Hirano in [13], a ring R is called a *quasi-Armendariz ring* if whenever $f(x)R[x]g(x) = 0$, where $f(x) = a_0 + a_1x + \cdots + a_kx^k$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$, it implies that $a_iRb_j = 0$ for all i, j. By Başer in [5], a module M_R is called a *quasi-Armendariz module* if whenever $m(x)R[x]f(x) = 0$, where $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ and $f(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$, it implies that $m_iRb_j = 0$ for all i, j.

Definition 3.1. *Let $\alpha : R \to R$ be an endomorphism. A module M_R is called α-skew quasi-Armendariz, if for any $m(x) = \sum_{i=0}^{k} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\alpha]$, $m(x)R[x;\alpha]f(x) = 0$ implies $m_iR\alpha^i(b_j) = 0$ for all i, j.*

Proposition 3.2. *Let α is an endomorphism of a ring R. The class of α-skew quasi-Armendariz modules are closed under direct sums, direct products and submodules.*
Put, $rAnn_R(sub(M)) = \{r_R(N) \mid N \text{ is submodule of } M\}$ and
$rAnn_{R[x;\alpha]}(sub(M[x;\alpha])) = \{r_{R[x;\alpha]}(V) \mid V \text{ is submodule of } M[x;\alpha]\}$.

Theorem 3.3. *Let M_R be an α-compatible module. Then the following statements are equivalent:*
(1) M_R is α-skew quasi Armendariz.
(2) $\phi' : rAnn_R(sub(M)) \to rAnn_{R[x;\alpha]}(sub(M[x;\alpha]))$ defined by
$\phi'(r_R(N)) = r_{R[x;\alpha]}(N) = r_{R[x;\alpha]}(N[x;\alpha])$ for every $r_R(N) \in rAnn_R(sub(M))$, is bijective.

Proof. (1) $\rightarrow$ (2) Let M_R be α-skew quasi Armendariz. We show ϕ' is surjective, because ϕ' is injective clearly. Let $r_{R[x;\alpha]}(V) \in rAnn_{R[x;\alpha]}(sub(M[x;\alpha]))$, for some submodule V of $M[x;\alpha]$. Then for $r_R(c_V R) \in rAnn_R(sub(M))$, we have $\phi'(r_R(c_V R)) = r_{R[x;\alpha]}(c_V R) = r_{R[x;\alpha]}(V)$. In fact, let $f(x) \in r_{R[x;\alpha]}(c_V R)$. So $c_V Rf(x) = 0$. In particular $c_V f(x) = 0$ and hence $V.f(x) = 0$, since M_R is α-compatible. So $f(x) \in r_{R[x;\alpha]}(V)$. Conversely let $g(x) = b_0 + b_1 x + \cdots + b_n x^n \in r_{R[x;\alpha]}(V)$, then $V.g(x) = 0$. Since V is a submodule of $M[x;\alpha]$, $VR[x;\alpha]g(x) = 0$. So $v(x)[x;\alpha]g(x) = 0$ for all $v(x) = v_0 + \cdots + v_\ell x^\ell \in V$. Since M_R is α-skew quasi Armendariz, $v_i R\alpha^i(b_j) = 0$ for each i, j. Hence $v_i R b_j = 0$, since M_R is α-compatible. So $c_V Rg(x) = 0$ and therefore $g(x) \in r_{R[x;\alpha]}(c_V R)$. Consequently ϕ' is surjective.
(2) $\rightarrow$ (1). Let $m(x) = m_0 + m_1 x + \cdots + m_k x^k \in M[x;\alpha]$ and $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x;\alpha]$ and $m(x)R[x;\alpha]f(x) = 0$. So $f(x) \in r_{R[x;\alpha]}(m(x)R[x;\alpha])$. On the other hand, $f(x) \in r_{R[x;\alpha]}(m(x)R[x;\alpha]) = r_{R[x;\alpha]}(N)[x;\alpha]$ for some submodule N of M, by hypothesis. So $a_j \in r_{R[x;\alpha]}(N) \subseteq r_{R[x;\alpha]}(N)[x;\alpha] = r_{R[x;\alpha]}(N[x;\alpha]) = r_{R[x;\alpha]}(m(x)R[x;\alpha])$. Hence we have $m(x)R[x;\alpha]a_j = 0$. Since M_R is α-compatible and $M[x;\alpha]$ is a right module over $R[x;\alpha]$, $m_i R\alpha^i(a_j) = 0$ for each i, j. Therefore M_R is α-skew quasi Armendariz.

Corollary 3.4. *Let M_R be an α-compatible module. Then the following statements are equivalent:*
(1) M_R is α-quasi Armendariz.
(2) $\phi' : rAnn_R(sub(M)) \rightarrow rAnn_{R[x;\alpha]}(sub(M[x;\alpha]))$ defined by $\phi'(r_R(N)) = r_{R[x;\alpha]}(N) = r_{R[x;\alpha]}(N[x;\alpha])$ for every $r_R(N) \in rAnn_R(sub(M))$, is bijective.

Corollary 3.5. ([6], Theorem 2.7) *Let M_R be an α-quasi Armendariz module. Then M_R is p.q.-Bear if and only if $M[x;\alpha]_{R[x;\alpha]}$ is p.q.-Bear.*

Corollary 3.6. ([6], Theorem 2.8) *Let M_R be a reduced module. Then the following statements are equivalent:*
(1) M_R is p.p.
(2) M_R is p.q.-Bear.
(3) $M[x]_{R[x]}$ is p.p.
(4) $M[x]_{R[x]}$ is p.q.-Bear.

Theorem 3.7. Let α be an endomorphism of a ring R. Then R is α-skew quasi-Armendariz if and only if every flat right R-module is α-skew quasi-Armendariz.

Proof. The proof is similar to that of [18, Theorem 2.11].

N. Agayev et al. [1], introduced and studied the notion of *abelian modules* as a generalization of abelian rings. A module M_R is called abelian if, for each $m \in M$, $a \in R$ and each idempotent $e \in R$, $mae = mea$. Hence R is abelian if and only if R_R is an abelian module. Every Armendariz module and hence every reduced module is abelian.

We end this section by proving that quasi-Armendariz modules are not abelian in general.

Example 3.8. Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. By [13, Corollary 3.15], R is a quasi-Armendariz ring, so R_R is a quasi-Armendariz module. But it is easy to see that R_R is not abelian.

4 Extensions of α-skew quasi-Armendariz modules

Let R be a ring and M be an (R, R)-bimodule. The trivial extension of R by M is defined to be the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}$, where $a \in R$ and $m \in M$. The endomorphism α of R is extended to $T(R, M)$ given by

$$\overline{\alpha}\left(\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & m \\ 0 & \alpha(a) \end{pmatrix}.$$

One can show that $T(R, M)[x; \overline{\alpha}] \cong T(R\,[x; \alpha]\,, M[x; \alpha])$.

Proposition 4.1. *Let M be an α-compatible (R, R)-bimodule. If the trivial extension $T(R, M)$ is α-skew quasi-Armendariz, then M is α-skew quasi-Armendariz.*

Proof. Let $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x; \alpha]$, $f(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x; \alpha]$ and $m\,(x)\,R\,[x; \alpha]\,f(x) = 0$. For each $a \in R$ and $m \in M$, we have the following equation:

$$\left(\sum_{i=0}^{k} \begin{pmatrix} 0 & m_i \\ 0 & 0 \end{pmatrix} x^i\right) \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \left(\sum_{j=0}^{n} \begin{pmatrix} b_j & 0 \\ 0 & b_j \end{pmatrix} x^j\right) =$$

$$\begin{pmatrix} 0 & m(x) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}\begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} = \begin{pmatrix} 0 & m\,(x)\,af(x) \\ 0 & 0 \end{pmatrix} = 0.$$

Since $T(R,M)$ is α-skew quasi-Armendariz, it implies that $\begin{pmatrix} 0 & m_i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}\overline{\alpha}^i\begin{pmatrix} b_j & o \\ 0 & b_j \end{pmatrix} = 0$ and so $m_i a\alpha^i\,(b_j) = 0$ for each i,j. Therefore $m_i R\alpha^i(b_j) = 0$ and the result follows.

Corollary 4.2. *Let R be an α-compatible ring. If the trivial extension $T(R,R)$ is α-skew quasi-Armendariz, then R is also α-skew quasi-Armendariz.*

Now we consider when the trivial extension $T(R,M)$ is α-skew quasi-Armendariz?

Theorem 4.3. *Let M be an (R,R)-bimodule and α-compatible such that:*

(i) R is an α-skew quasi-Armendariz ring;

(ii) M is an α-skew quasi-Armendariz module and Armendariz left R-module;

(iii) If $f\,(x)\,R\,[x;\alpha]\,g\,(x) = 0$, then $f\,(x)\,M[x;\alpha] \cap M[x;\alpha]g(x)$=0.

Then the trivial extension $T(R,M)$ is an α-skew quasi-Armendariz ring.

Proof. Suppose that $A(x)T\,(R,M)\,[x;\overline{\alpha}]\,B(x) = 0$, where

$A(x) = \begin{pmatrix} a_0 & m_0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 & m_1 \\ 0 & a_1 \end{pmatrix} x + \cdots + \begin{pmatrix} a_s & m_s \\ 0 & a_s \end{pmatrix} x^s$,

$B(x) = \begin{pmatrix} b_0 & n_0 \\ 0 & b_0 \end{pmatrix} + \begin{pmatrix} b_1 & n_1 \\ 0 & b_1 \end{pmatrix} x + \cdots + \begin{pmatrix} b_t & n_t \\ 0 & b_t \end{pmatrix} x^t \in T(R,M)[x;\overline{\alpha}]$. Let

$f(x) = a_0 + a_1x + \cdots + a_sx^s$, $g(x) = b_0 + b_1x + \cdots + b_tx^t \in R[x;\alpha]$,

$m(x) = m_0 + m_1x + \cdots + m_sx^s$, $n(x) = n_0 + n_1x + \cdots + n_tx^t \in M[x;\alpha]$. For an arbitrary $\begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \in T(R,M)$, it follows that

$0 = \begin{pmatrix} f(x) & m(x) \\ 0 & f(x) \end{pmatrix}\begin{pmatrix} a & m \\ 0 & a \end{pmatrix}\begin{pmatrix} g(x) & n(x) \\ 0 & g(x) \end{pmatrix} =$

$\begin{pmatrix} f(x)ag(x) & f(x)an(x) + f(x)mg(x) + m(x)ag(x) \\ 0 & f(x)ag(x) \end{pmatrix}$. Hence $f(x)ag(x) = 0$ and

$f(x)an(x) + f(x)mg(x) + m(x)ag(x) = 0$. So $a_iR\alpha^i(b_j) = 0$ for all i,j, as R is α-skew quasi-Armendariz. Since $f(x)(an(x) + mg(x)) + m(x)ag(x) = 0$, $f(x)(an(x) + mg(x)) = -m(x)ag(x) \in f\,(x)\,M[x;\alpha] \cap M[x;\alpha]g(x) = 0$, so $f(x)(an(x)+mg(x)) = 0 = m(x)ag(x)$. Since $a \in R$ is arbitrary, $m(x)Rg(x) = 0$, so $m_iR\alpha^i\,(b_j) = 0$, for all i,j, as M is α-skew quasi-Armendariz. Also we have $f(x)an(x) = -f(x)mg(x) \in f\,(x)\,M\,[x;\alpha] \cap M\,[x;\alpha]\,g\,(x) = 0$. Thus $f(x)an(x) = 0$ and so $a_ian_j = 0$, by (ii). We have $f\,(x)\,mg\,(x) = 0$ and $f\,(x)\,m \in M[x;\alpha]$, so by (ii) we have $a_im\alpha^i\,(b_j) = 0$ for all i,j. Thus

$\begin{pmatrix} a_i & m_i \\ 0 & a_i \end{pmatrix} \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \overline{\alpha}^i \begin{pmatrix} b_j & n_j \\ 0 & b_j \end{pmatrix} =$
$\begin{pmatrix} a_i a\alpha^i(b_j) & a_i a n_j + a_i m\alpha^i(b_j) + m_i a\alpha^i(b_j) \\ 0 & a_i a\alpha^i(b_j) \end{pmatrix} = 0$ for all i, j and each $\begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \in T(R, M)$. Therefore the trivial extension $T(R, M)$ is α-skew quasi-Armendariz.

Corollary 4.4. *Let R be an α-compatible ring such that*
(1) R is an α-skew quasi-Armendariz ring.
(2) If $f(x) R[x;\alpha] g(x) = 0$, then $f(x) R[x;\alpha] \cap R[x;\alpha] g(x) = 0$.
Then the trivial extension $T(R, R)$ is an α-skew quasi-Armendariz ring.

REFERENCES

[1] N. Agayev, G. Gungoroglu, A. Harmanci and S. Halicioglu, Abelian modules, *Acta Math. Univ. Comenianae* **78**(**2**)(**2009**), 235-244.
[2] D.D. Anderson and V. Camillo, Armendariz rings and Gaussian rings, *Comm. Algebra***26**(7) (**1998**) 2265-2275.
[3] S. Annin, Associated and Attached Primes Over Noncommutative Rings, *phd Thesis,* University of California at Berkeley (**2002**).
[4] E.P. Armendariz, A note on extensions of Baer and p.p.-rings, *J. Austral. Math. Soc.* **18** (**1974**) 470-473.
[5] M. Başer, On Armendariz and quasi-Armendariz modules, *Note di Matematica* n. 1 **26** (**2006**), 173-177.
[6] M. Başer and M.T. Koşan, On quasi-Armendariz modules, *Taiwan. J. of Math.* Vol. **12**, No. **3**, (**2008**) 573-582.
[7] G.F. Birkenmeier, J.Y. Kim and J.K. Park, On quasi-Baer rings, *Contemp. Math.* **259** (**2000**), 67-92.
[8] G.F. Birkenmeier, J.Y. Kim and J.K. Park, Principally quasi-Baer rings, *Comm. Algebra* **29**(2) (**2001**) 639-660.
[9] G.F. Birkenmeier, J.Y. Kim and J.K. Park, Polynomial extensions of Baer and quasi-Baer rings, *J. Pure Appl. Algebra* **159** (**2001**) 24-42.
[10] G.F. Birkenmeier, J.Y. Kim and J.K. Park, On polynomial extensions of principally quasi-Baer rings, *Kyungpook Math. J.* **40** (**2000**) 247 253.
[11] W.E. Clark, Twisted matrix units semigroup algebras, *Duke Math. J.* **34** (**1967**) 417-424.
[12] E. Hashemi and A. Moussavi, Polynomial extensions of quasi-Baer rings. *Acta Math. Hungar.* **107** (**2005**), no. 3, 207-224.
[13] Y. Hirano, On annihilator ideals of a polynomial ring over a noncommutative ring, *J. Pure Appl. Algebra* **168** (**2002**), 45-52.
[14] C.Y. Hong, T.K. Kwak, S.T. Rizvi, Extensions of generalized Armendariz

rings, *Algebra Colloq.(13)* **2** (**2006**), 253-266.
[15] I. Kaplansky, *Rings of Operators, Benjamin, New York,* **(1965)**.
[16] T.K. Lee and Y. Zhou, Reduced Modules, *Rings, Modules, Algebras, and Abelian Groups, Lecture Notes in Pure and Appl. Math.*, vol. **236**, Marcel Dekker, New York, **(2004)**, pp. 365-377.
[17] P. Pollingher and A. Zaks, On Baer and quasi-Baer rings, *Duke Math. J.* **37**(**1970**), 127-138.
[18] C.P. Zhang and J.L. Chen, α-Skew Armendariz modules and α-Semicommutative modules, *Taiwan. J. of Math.* Vol. **12**, No. **2**, (**2008**) 473-486.

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