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On a Novel Approach to Decompose Finite Energy Functions by Energy Operators and its Application to the General Wave Equation

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Abstract

This work aims at introducing some energy operators linked to the Teager-Kaiser energy operator and expands it from time to space in the real domain. We then show the decomposition of the functions of space and time differentiable at least twice in $\mathbb{R}$ using those energy operators. The second part of the work focuses on using the energy operators to redefine the general wave equation. The method is first established for the wave equation in a non-dispersive medium and then extended for a particular case of the wave equation in a dispersive medium. Through the method, the author defines the energy quantities Σ^- and Σ^+ associated for a given solution of the wave equation (f). An important property (Property 3) shows that there are unique energy quantities (Σ^-, Σ^+) associated with the absolute value of a given solution of the wave equation. Some examples are used to look into the theory.

Mathematics Subject Classification: 35-02

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1 Introduction

The Teager-Kaiser energy operator has been introduced in [Kaiser (1990)] and the past decade has seen numerous applications flourishing in signal processing [Bovik et al. (1993)],[Mitra et al. (1991)] and [Lin et al. (1995)]; speech recognition [Kaiser et al. (1993)] and more exotic works such as mobile phone positioning [Hamila et al. (1999)] and mammals localization [Kandia et al. (2006)].

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The presented work is not based on the current main stream research work surrounding this operator. By extending the general definition in the real domain of the Teager-Kaiser operator defined in [Kaiser (1990)] or [Kaiser (1993)] in time and in space, the author then redefines the wave equation as a linear sum of energy operators. The interest of this early work is to define the energy operators and show the possible applications to the wave theory and which may then contribute further.

2 Teager-Kaiser Energy Operators: a brief overview

Following the general description in [Kaiser (1990)], the general formula of the operator can be written as Ψ_c a quadratic form of $\mathbb{R}\times\ \mathbb{R} -> \mathbb{R}$ defined in the real domain as:

$$\Psi_c[X(t), Y(t)] = \frac{dX(t)}{dt}\frac{dY(t)}{dt} - 0.5[\frac{d^2X(t)}{dt^2}Y(t) + \frac{d^2Y(t)}{dt^2}X(t)] \qquad (1)$$

Where X and Y are two continuous functions in $\mathbb{R} -> \mathbb{R}$ and at least twice differentiable. Note that in [Hamila et al. (1999b)], the general formula stated in the Equation (1) is defined in the complex domain $\mathbb{C}$. For the purpose of simplicity, only the definition of the operator in the real domain is used throughout the present work.
The bilinearity of the function Ψ_c: $\mathbb{C} \times \mathbb{C} -> \mathbb{C}$ in the complex domain is shown in [Boudraa et al. (2009)]. Thus, the property is obviously applied to the real domain Ψ_c: $\mathbb{R} \times \mathbb{R} -> \mathbb{R}$. For the remainder of this article, let us rename Ψ_c as Ψ_R^-.

3 Extension to another Energy Operator

Now let us define Ψ_R^+: $\mathbb{R} \times \mathbb{R} -> \mathbb{R}$ defined in the real domain as

$$\Psi_R^+[X(t), Y(t)] = \frac{dX(t)}{dt}\frac{dY(t)}{dt} + 0.5[\frac{d^2X(t)}{dt^2}Y(t) + \frac{d^2Y(t)}{dt^2}X(t)] \qquad (2)$$

Where X and Y are two continuous functions in $\mathbb{R} -> \mathbb{R}$ and twice differentiable. One can show that Ψ_R^+ is a bilinear form with commutativity ($\Psi_R^+[X(t), Y(t)] = \Psi_R^+[Y(t), X(t)]$) in $\mathbb{R}$.

Property 1: All functions which verify the bilinearity properties of the energy operators Ψ_R^+ and Ψ_R^- have the square of their first derivative equal to the sum of the operators.

Proof. Let us chose a function X from $\mathbb{R} -> \mathbb{R}$ and twice differentiable. Thus, one can write:

$$\Psi_R^+[X(t), X(t)] = \frac{dX(t)}{dt}\frac{dX(t)}{dt} + 0.5[\frac{d^2X(t)}{dt^2}X(t) + \frac{d^2X(t)}{dt^2}X(t)]$$
$$\Psi_R^-[X(t), X(t)] = \frac{dX(t)}{dt}\frac{dX(t)}{dt} - 0.5[\frac{d^2X(t)}{dt^2}X(t) + \frac{d^2X(t)}{dt^2}X(t)]$$

And then,

$$\Psi_R^+[X(t), X(t)] + \Psi_R^-[X(t), X(t)] = \frac{dX(t)}{dt}\frac{dX(t)}{dt}$$

The notation $\Psi_R^+[X]$ standing for $\Psi_R^+[X(t), X(t)]$ is sometimes used in the sequel. Furthermore, $\Psi_R^+[X]$ will be noted as $\Psi_R^{t+}[X]$ to underline the dependence in *time*. □

Note if $X(t)$ is a real deterministic signal at time t such as $x(n)$, following [Lin et al. (1995)] the energy operator $\Psi_R^-[x(n)]$ is then redefined as:

$$\Psi_R^-[x(n)] = x(n)^2 - x(n+1)x(n-1) \tag{3}$$

Thus, in this case, we have

$$\Psi_R^+[x(n)] + \Psi_R^-[x(n)] = x(n)^2$$

Which means that in the case of a deterministic signal, the sum of the two operators is equal to the square of the signal. Further development yields to the general definition of the energy of the signal by taking the mean as for example in:

$$E\{\Psi_R^+[x(n)] + \Psi_R^-[x(n)]\} = E\{x(n)^2\}$$

4 Extension of the definition from time to space

In the previous sections, the energy operators Ψ_R^+ and Ψ_R^- were applied to a functions X: $\mathbb{R} -> \mathbb{R}$ continuous and twice differentiable in **time**. Now, if one may consider the case of the function α: $\mathbb{R}->\mathbb{R}$ continue and twice differentiable in **space**. Then, one may define a similar energy operator in space, $\Psi_R^{s-}[\alpha]$ (with s stands for space in the notation), in the Cartesian referential [x, y, z] such as for the one dimensional case:

$$\Psi_R^{s\{dx\}-}[\alpha] = \frac{d\alpha(x)}{dx}\frac{d\alpha(x)}{dx} - 0.5[\frac{d^2\alpha(x)}{dx^2}\alpha(x) + \frac{d^2\alpha(x)}{dx^2}\alpha(x)] \tag{4}$$

with $\Psi_R^{s\{dx\}-}$, the energy operator in space developed following the x Cartesian coordinate (dx in the notation). To extend the above definition of the energy operator in the 3 dimensional space, one has to construct the operator to conserve the bilinearity properties stated in the first paragraph. One way is then to define the energy operator in 3D (Ψ_R^{s-}) as a sum of one dimensional space energy operators. Let be a real function twice differentiable α: $\mathbb{R}^3 -> \mathbb{R}$ or shortened $\alpha(x, y, z)$ in the Cartesian referential. Note that sometimes we write only α in the following equations such as $\alpha := \alpha(x, y, z)$. When we apply the energy operator Ψ_R^{s-} on α such as:

$$\Psi_R^{s-}[\alpha] = \Psi_R^{s\{dx\}-}[\alpha] + \Psi_R^{s\{dy\}-}[\alpha] + \Psi_R^{s\{dz\}-}[\alpha]$$

Using the definition of the operators stated in Equation (4), it allows:

$$\begin{aligned}\Psi_R^{s-}[\alpha(x,y,z)] &= (\frac{\partial\alpha}{\partial x})^2 + (\frac{\partial\alpha}{\partial y})^2 + (\frac{\partial\alpha}{\partial z})^2 \\ &\quad -0.5([\frac{\partial^2\alpha}{\partial x^2}\alpha + \frac{\partial^2\alpha}{\partial x^2}\alpha] + [\frac{\partial^2\alpha}{\partial y^2}\alpha + \frac{\partial^2\alpha}{\partial y^2}\alpha] + \\ &\quad [\frac{\partial^2\alpha}{\partial z^2}\alpha + \frac{\partial^2\alpha}{\partial z^2}\alpha])\end{aligned}$$

And then,

$$\Psi_R^{s-}[\alpha(x,y,z)] = \sum_{i=1}^{3}(\frac{\partial\alpha}{\partial l_i})^2 - \alpha\frac{\partial^2\alpha}{\partial l_i^2} \quad (5)$$

Note that in Equation (5), l_i means the X-axis in the Cartesian referential if $i = 1$. One step further is to consider not α only as a function of space and at least twice differentiable, but associated with the Cartesian referential defined with the standard unitary vector $\{\vec{i}, \vec{j}, \vec{k}\}$. It then allows to define the gradiant ($\vec{\nabla}$) and Laplacian operator ($\triangle$) of the real function α twice differentiable such as:

$$\vec{\nabla}\alpha = \frac{\partial\alpha}{\partial x}\times\vec{i} + \frac{\partial\alpha}{\partial y}\times\vec{j} + \frac{\partial\alpha}{\partial z}\times\vec{k} \quad (6)$$

$$\triangle\alpha = \nabla.\nabla\alpha = \frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} + \frac{\partial^2\alpha}{\partial z^2} \quad (7)$$

Note that (.) in the above equation is a scalar product between the nabla operator. Thus, the definitions given in the Equations (6) help to rewrite the general definition of the energy operator in space Ψ_R^{s-} in Equation (8) such as:

$$\Psi_R^{s-}[\alpha] = \vec{\nabla}\alpha.\vec{\nabla}\alpha - \alpha\triangle\alpha \quad (8)$$

In the same way that Ψ_R^{s-} has been defined in one dimension and then generalized to three dimensions, the space energy operator Ψ_R^{s+} can be then defined in one dimension and $3D$ such as:

$$\Psi_R^{s\{dx\}+}[\alpha] \quad = \quad \frac{d\alpha(x)}{dx}\frac{d\alpha(x)}{dx} + \frac{d^2\alpha(x)}{dx^2}\alpha(x) \tag{9}$$

$$\Psi_R^{s+}[\alpha] \quad = \quad \vec{\nabla}\alpha.\vec{\nabla}\alpha + \alpha\triangle\alpha \tag{10}$$

5 Redefinition of the General Wave Equation

5.1 The wave equation in a non-dispersive medium

From [Pain (2005)], the general formula of a wave equation in a non-dispersive medium (e.g. free space) can be written in a one dimensional case as:

$$\frac{\partial^2\gamma}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\gamma}{\partial t^2} \tag{11}$$

With $\gamma(x,t)$ the general solution of the equation and c the speed of the wave in the given medium (e.g. in free space, c is the speed of light). Using the Equations in (3), (4) and (8), it is then possible to transform the wave equation by multiplying the two sides with the general solution as follow:

$$\begin{aligned} \gamma\frac{\partial^2\gamma}{\partial x^2} &= 0.5[\Psi_R^{s\{dx\}+}(\gamma) - \Psi_R^{s\{dx\}-}(\gamma)] \\ \frac{\gamma}{c^2}\frac{\partial^2\gamma}{\partial t^2} &= \frac{1}{2c^2}[\Psi_R^{t+}(\gamma) - \Psi_R^{t-}(\gamma)] \end{aligned} \tag{12}$$

Which leads to rewrite the wave equation as:

$$\begin{aligned} [\Psi_R^{s\{dx\}+} - \Psi_R^{s\{dx\}-}](\gamma) &= \frac{1}{c^2}[\Psi_R^{t+} - \Psi_R^{t-}](\gamma) \\ \Psi_R^{s\{dx\}+}(\gamma) - \frac{1}{c^2}\Psi_R^{t+}(\gamma) &= \Psi_R^{s\{dx\}-}(\gamma) - \frac{1}{c^2}\Psi_R^{t-}(\gamma) \end{aligned} \tag{13}$$

The last equation can also be seen as:

$$\Sigma^+(\gamma) \quad = \quad \Sigma^-(\gamma) \tag{14}$$

The redefinition of the wave equation with the energy operators underlines both the split of the energy in space and in time by using the family of energy operators defined in the previous sections. Note that, it is interesting to see the formation of two energy quantities Σ^{+} and Σ^{-} in Equation (14). In a non-dispersive medium those two quantities are equal and unified the energy in time and in space for a given wave.

Property 2: In a non- dispersive medium, the energy quantities Σ^{+} and Σ^{-} are equal for each solution.

Proof. Let us use the general solution of a wave equation which is the sum of two waves traveling in the opposite direction in space at a given time following [Pain (2005)]. Then if γ is the general solution of the Equation (11), it can be redefined in one dimensional space as:

$$\gamma(x,t) \;=\; f(t-x/c)+g(t+x/c)$$

And

$$\begin{aligned}
\gamma\frac{\partial^2\gamma}{\partial x^2} &= f\frac{\partial^2 f}{\partial x^2}+g\frac{\partial^2 g}{\partial x^2}+(f\frac{\partial^2 g}{\partial x^2}+g\frac{\partial^2 f}{\partial x^2})\\
\gamma\frac{\partial^2\gamma}{\partial t^2} &= f\frac{\partial^2 f}{\partial t^2}+g\frac{\partial^2 g}{\partial t^2}+(f\frac{\partial^2 g}{\partial t^2}+g\frac{\partial^2 f}{\partial t^2})
\end{aligned}$$

Following the same development given in the Equation (12) leads to:

$$\begin{aligned}
\gamma\frac{\partial^2\gamma}{\partial x^2} &= 0.5[\Psi_R^{s\{dx\}+}(f)-\Psi_R^{s\{dx\}-}(f)]\\
&\quad +0.5[\Psi_R^{s\{dx\}+}(g)-\Psi_R^{s\{dx\}-}(g)]+[\Psi_R^{s\{dx\}+}(f,g)-\Psi_R^{s\{dx\}-}(f,g)]
\end{aligned}$$

$$\begin{aligned}
\gamma\frac{\partial^2\gamma}{\partial t^2} &= 0.5[\Psi_R^{t+}(f)-\Psi_R^{t-}(f)]+0.5[\Psi_R^{t+}(g)-\Psi_R^{t-}(g)]+[\Psi_R^{t+}(f,g)-\Psi_R^{t-}(f,g)]\\
\gamma\frac{\partial^2\gamma}{\partial x^2}-\frac{\gamma}{c^2}\frac{\partial^2\gamma}{\partial t^2} &= 0.5[\Sigma^{+}(f)-\Sigma^{-}(f)]+0.5[\Sigma^{+}(g)-\Sigma^{-}(g)]+[\Sigma^{+}(f,g)-\Sigma^{-}(f,g)]
\end{aligned}$$

Finally the last equation is nullified (as it is in a non-dispersive medium) and the result is:

$$0.5[\Sigma^{+}(f)-\Sigma^{-}(f)]+0.5[\Sigma^{+}(g)-\Sigma^{-}(g)]+[\Sigma^{+}(f,g)-\Sigma^{-}(f,g)]=0 \quad (15)$$

Which means if $\forall\ (x,t)\ f(x,t)\neq g(x,t)$,

$$\left\{\begin{array}{lcl}
\Sigma^{+}(f) &=& \Sigma^{-}(f)\\
\Sigma^{+}(g) &=& \Sigma^{-}(g)\\
\Sigma^{+}(f,g) &=& \Sigma^{-}(f,g)
\end{array}\right. \quad (16)$$

□

Furthermore the Equation (16) shows also that the cross-energy terms $\Sigma^+(f,g)$ and $\Sigma^-(f,g)$ are also equal. Thus this proof also underlines a symmetry in the energy quantities Σ^+ and Σ^- for the functions f, g and the cross-energy term in the specific case of the non-dispersive wave equation. In addition in Equation (15), it is important to underline the separation between the Energy Operators in time and space. This separation may be carefully viewed as a separation of the energy in time and the energy in space of the total energy of the general solution of the wave equation. Here, the two quantities are equal for the special case of the non-dispersive medium. By using the energy operators family, it may be then stated that the total energy associated with the two waves solutions of the the wave equation in a non-disspersive medium is splitted into three energy quantities (Σ^+(f), Σ^-(f)), (Σ^+(g), Σ^-(g)) and (Σ^+(f,g), Σ^-(f,g)).

By using the Equation (10), the generalisation in 3D of the redefinition of the wave equation leads for a Cartesian referential with

$$\forall (x,y,z,t) \in \mathbb{R}^4 \qquad \gamma(x,y,z,t) \tag{17}$$

Then,

$$\begin{aligned} \gamma \triangle \gamma &= 0.5[\Psi_R^{s+}(\gamma) - \Psi_R^{s-}(\gamma)] \\ \frac{\gamma}{c^2}\frac{\partial^2 \gamma}{\partial t^2} &= \frac{1}{2c^2}[\Psi_R^{t+}(\gamma) - \Psi_R^{t-}(\gamma)] \end{aligned} \tag{18}$$

And then,

$$\begin{aligned} \Sigma_{3D}^+(\gamma) &= \Psi_R^{s+}(\gamma) - \frac{1}{c^2}\Psi_R^{t+}(\gamma) \\ \Sigma_{3D}^-(\gamma) &= \Psi_R^{s-}(\gamma) - \frac{1}{c^2}\Psi_R^{t-}(\gamma) \end{aligned} \tag{19}$$

Σ_{3D}^+ and Σ_{3D}^- are respectively the generalisation in 3D of the quantity Σ^+ and Σ^-.

Property 3: Σ^+ and Σ^- are two unique energy quantities for a given solution of the wave equation as described in (11).

Proof. Let us take two solutions f_1 and f_2 of the wave equation such as:

$$\forall (x,t) \in \mathbb{R}^2 | \qquad f_1(x,t) \qquad and \qquad f_2(x,t) \;\; \neq \;\; 0$$

We will limit the proof to the 1D case, as it is easily extended to the 3D case. Based on the definition of the energy quantities in Equation (14), one can write

:

$$\begin{aligned}\Sigma^+(f_1) &= \Sigma^+(f_2)\\ \Sigma^-(f_1) &= \Sigma^-(f_2)\end{aligned} \tag{20}$$

The development using the definition in Equation (13) can be then summarized as

$$\begin{aligned}\Psi_R^{t+}(f_1) - \Psi_R^{t+}(f_2) &= 0\\ \Psi_R^{s\{dx\}+}(f_1) - \Psi_R^{s\{dx\}+}(f_2) &= 0\\ \Psi_R^{t-}(f_1) - \Psi_R^{t-}(f_2) &= 0\\ \Psi_R^{s\{dx\}-}(f_1) - \Psi_R^{s\{dx\}-}(f_2) &= 0\end{aligned} \tag{21}$$

Going further in the development using the general definitions (Equations (1) and (2)),

$$\begin{aligned}\frac{\partial f_1(x,t)}{\partial x}\frac{\partial f_1(x,t)}{\partial x} - \frac{\partial^2 f_1(x,t)}{\partial x^2}f_1(x,t) &= \frac{\partial f_2(x,t)}{\partial x}\frac{\partial f_2(x,t)}{\partial x} - \frac{\partial^2 f_2(x,t)}{\partial x^2}f_2(x,t)\\ \frac{\partial f_1(x,t)}{\partial t}\frac{\partial f_1(x,t)}{\partial t} - \frac{\partial^2 f_1(x,t)}{\partial t^2}f_1(x,t) &= \frac{\partial f_2(x,t)}{\partial t}\frac{\partial f_2(x,t)}{\partial t} - \frac{\partial^2 f_2(x,t)}{\partial t^2}f_2(x,t)\end{aligned}$$

After some transformations,

$$\begin{aligned}(\frac{\partial f_1(x,t)}{\partial x})^2 - (\frac{\partial f_2(x,t)}{\partial x})^2 &= 0\\ (\frac{\partial f_1(x,t)}{\partial t})^2 - (\frac{\partial f_2(x,t)}{\partial t})^2 &= 0\\ \frac{\partial^2 f_2(x,t)}{\partial t^2}f_2(x,t) - \frac{\partial^2 f_1(x,t)}{\partial t^2}f_1(x,t) &= 0\\ \frac{\partial^2 f_2(x,t)}{\partial x^2}f_2(x,t) - \frac{\partial^2 f_1(x,t)}{\partial x^2}f_1(x,t) &= 0\end{aligned} \tag{22}$$

And further transformations based on the linearity of the derivative,

$$\begin{aligned}(\frac{\partial f_1(x,t)}{\partial x} - \frac{\partial f_2(x,t)}{\partial x})(\frac{\partial f_1(x,t)}{\partial x} + \frac{\partial f_2(x,t)}{\partial x}) &= 0\\ (\frac{\partial f_1(x,t)}{\partial t} - \frac{\partial f_2(x,t)}{\partial t})(\frac{\partial f_1(x,t)}{\partial t} + \frac{\partial f_2(x,t)}{\partial t}) &= 0\end{aligned}$$

For the second pairs of equalities in the Equation (22), it is necessary to show that

$$0.5\frac{\partial^2 (f_2(x,t))^2}{\partial x^2} = \frac{\partial^2 f_2(x,t)}{\partial x^2}f_2(x,t) + (\frac{\partial f_2(x,t)}{\partial x})^2 \tag{23}$$

By combining Equations (22) and (23), it leads to:

$$\begin{aligned}\frac{\partial^2(f_2(x,t))^2}{\partial x^2}-\frac{\partial^2(f_1(x,t))^2}{\partial x^2} &= 0\\ \frac{\partial^2(f_2(x,t))^2}{\partial t^2}-\frac{\partial^2(f_1(x,t))^2}{\partial t^2} &= 0\end{aligned} \quad (24)$$

And the results of Equations (23) and (24) show that:

$$\forall(x,t)\in R^2 f_1(x,t)=\pm f_2(x,t) \quad (25)$$

To conclude, if the energy quantities Σ^+ and Σ^- are equal for two solutions of the non-dispersive wave equation, then the solutions are equal in absolute value. Thus, each solution of the wave solution in the non-dispersive case, can be associated in absolute value to an unique energy quantity (Σ^+, Σ^-). □

Finally, one can see that the proof starts from the Equation (20) and this is independent of which medium the wave are traveling into (dispersive or non-dispersive). Thus, Property 3 is easily expanded to any medium.

5.2 An example for the non-dispersive wave equation

Let us see the development of the equations given in the section above, if the theory is applied to f a simple harmonic displacement of an oscillator at position x (one dimensional case) and time t as in [Pain (2005)] solution of the general wave equation in a non-dispersive medium.

$$f(x,t)=a\sin\left(\frac{2\pi}{\lambda}(ct-x)\right) \quad (26)$$

With λ the wavelength, c the speed of light and a the amplitude of the solution ($a\in R$). Note that $\frac{2\pi}{\lambda}$ is shortened as l in the following. Following Equations (1), (2) and (4), then

$$\begin{aligned}\Psi_R^{t+}(f) &= (lca)^2(\cos^2(l(ct-x))-\sin^2(l(ct-x)))\\ \Psi_R^{s+}(f) &= (la)^2(\cos^2(l(ct-x))-\sin^2(l(ct-x)))\\ \Psi_R^{t-}(f) &= (lca)^2\\ \Psi_R^{s-}(f) &= (la)^2\end{aligned}$$

And then,

$$\begin{aligned}\Sigma^+(f) &= 0\\ \Sigma^-(f) &= 0\end{aligned} \quad (27)$$

If then, the opposite waves is also taken into account as:

$$g(x,t) = a\sin\left(\frac{2\pi}{\lambda}(ct+x)\right) \tag{28}$$

The results are identical to Equation (27):

$$\begin{aligned} \Sigma^{+}(g) &= 0 \\ \Sigma^{-}(g) &= 0 \end{aligned} \tag{29}$$

However, looking at the cross-energy terms defined in Equation (15), the numerical results are:

$$\begin{aligned} \Sigma^{+}(f,g) &= -2(la)^2(\cos\left(\frac{2\pi}{\lambda}(ct-x)\right)\cos\left(\frac{2.\pi}{\lambda}(ct+x)\right)) \\ \Sigma^{-}(f,g) &= -2(la)^2(\cos\left(\frac{2\pi}{\lambda}(ct-x)\right)\cos\left(\frac{2\pi}{\lambda}(ct+x)\right)) \end{aligned} \tag{30}$$

5.3 The wave equation in a dispersive medium

All the previous paragraph is based on the hypothesis if the medium is non-dispersive. Let us then take an example when the right-hand of the equation is equal to a dispersive term of the form $\frac{1}{d}\frac{\partial\gamma(s,t)}{\partial t}$ with d a given value in the real domain $\mathbb{R}$ not null. The argument s of γ replaces the coordinates in space. The assumption behind this type of equation is the energy losses are small when the wave propagate in time and space. This is why the equation is a combination of wave and diffusion equations. The wave equation is then equal to:

$$\frac{\partial^2\gamma(s,t)}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2\gamma(s,t)}{\partial t^2} = \frac{1}{d}\frac{\partial\gamma(s,t)}{\partial t} \tag{31}$$

Applying the same method to transform the wave equation as described in Equation (12) shows:

$$0.5[\Sigma^{+}(\gamma)(s,t) - \Sigma^{-}(\gamma)(s,t)] = \frac{\gamma(s,t)}{d}\frac{\partial\gamma(s,t)}{\partial t} \tag{32}$$

In this case, the Property 2 stated in the previous section has to be revisited for the dispersive medium. By using the full solution of the wave equation $\gamma(s,t) = f(s,t) + g(s,t)$ and following the development in Equation (12), then

$$\begin{aligned} & 0.5[\Sigma^{+}(f(s,t)) - \Sigma^{-}(f(s,t))] \\ & +0.5[\Sigma^{+}(g(s,t)) - \Sigma^{-}(g(s,t))] \\ & +[\Sigma^{+}(f(s,t),g(s,t)) - \Sigma^{-}(f(s,t),g(s,t))] \\ & \qquad = \frac{(f(s,t)+g(s,t))}{d}\frac{\partial(f(s,t)+g(s,t))}{\partial t} \end{aligned} \tag{33}$$

The left-hand side of the equation can be factorized as follow,

$$\frac{(f(s,t)+g(s,t))}{d}\frac{\partial(f(s,t)+g(s,t))}{\partial t} = \frac{1}{2d}\frac{\partial(f(s,t)+g(s,t))^2}{\partial t}$$

By splitting the Equation (33), it leads to:

$$\begin{aligned} 0.5[\Sigma^+(f(s,t)) - \Sigma^-(f(s,t))] &= \frac{1}{2d}\frac{\partial(f(s,t))^2}{\partial t} \\ 0.5[\Sigma^+(g(s,t)) - \Sigma^-(g(s,t))] &= \frac{1}{2d}\frac{\partial(g(s,t))^2}{\partial t} \\ \Sigma^+(f(s,t),g(s,t)) - \Sigma^-(f(s,t),g(s,t)) &= \frac{1}{d}\frac{\partial(f(s,t).g(s,t))}{\partial t} \end{aligned} \quad (34)$$

One may see that there is now a relationship between Σ^+ and Σ^- which in this case is a function of time and space. In the special case of the wave equation in a dispersive medium, it is interesting to see that the right-hand side of the Equation (34) are product of functions (or square) and it then eliminates the positivity or negativity of the function. For a graphical interpretation, one may look at the function in the referential $(0, \Sigma^+, \Sigma^-)$.

It is important to remember Property 3 given the uniqueness of Σ^+ and Σ^- for a given solution γ of the wave equation, no matter the medium where the waves are propagating into.

5.4 An example of the wave equation in a dispersive medium in one dimension

From [Pain (2005)], a possible solution of the Equation (31) is :

$$\forall(x,t)\in\mathbb{R}^2 \qquad \gamma_1(x,t) = \gamma_m \exp(-a.x)\cos(\omega t - k.x) \quad (35)$$

Which is a wave with an amplitude attenuated while traveling in space. Note γ_m is the amplitude in $\mathbb{R}$, k is the wave number, c the speed of light and ω the pulsation such as [Pain (2005)]:

$$k^2 - a^2 = \omega^2/c^2, \text{with} \qquad a << k, \quad (36)$$

Using the general formula of the Energy Operators given in Equations (3) and (9), it results:

$$\begin{aligned} \Psi_R^{t+}(\gamma_1(x,t)) &= \omega^2\gamma_m^2\exp(-2ax)[\sin^2(ba) - \cos^2(ba)] \\ \Psi_R^{t-}(\gamma_1(x,t)) &= \omega^2\gamma_m^2\exp(-2ax) \\ \Psi_R^{s+}(\gamma_1(x,t)) &= 2a^2\gamma_1^2(x,t) + k^2\gamma_m^2\exp(-2ax)[\sin^2(ba) - \cos^2(ba)] \\ &\quad -4ak\gamma_m^2\exp(-2ax)\sin(ba)\cos(ba) \\ \Psi_R^{s-}(\gamma_1(x,t)) &= k^2\gamma_m^2\exp(-2ax) \end{aligned}$$

To write the above equations in a handy way, the following notation is used $ba = \omega t - k.x$. Thus, it is then possible to give the expression of the energy quantities:

$$\begin{aligned}\Sigma^+(\gamma_1(x,t)) &= 2a^2\gamma_1^2(x,t) - 4ak\gamma_m^2 \exp(-2ax)\sin(ba)\cos(ba) \\ \Sigma^-(\gamma_1(x,t)) &= 0\end{aligned}$$

But with the Equation (32), then

$$\begin{aligned}2a^2\gamma_1^2(x,t) - 4ak\gamma_m^2 \exp(-2ax)\sin(ba).\cos(ba) &= -\frac{\omega}{d}\gamma_m^2 \exp(-2ax)\sin(ba)\cos(ba) \\ 2a^2\cos(ba) - 4ak\sin(ba) &= -\frac{\omega}{d}\sin(ba) \\ \tan(\omega t - kx) &= 2a^2[4ak - \frac{\omega}{d}]^{-1}\end{aligned}$$

The last equation states that $d \neq \frac{c}{(4.a)}$. This result gives the boundaries to establish the special form of the wave equation (e.g. Equation (31)) and thereby the validity of the relationship between the energy quantities shown Equation (32).

6 Conclusions

This work has introduced a new family of operators called energy operators following the previous work on the Teager-Kaiser energy operator. This family of operators are defined in time and space (3D Cartesian coordinates). It then allows a new approach on the decomposition of the functions of space and time differentiable at least twice in $\mathbb{R}$. The second part of the work is focused on using the energy operators to redefine the general wave equation. The method is developed for the wave equation in a non-dispersive medium. When developing the method step-by-step, the energy quantities are then defined as Σ^- and Σ^+ associated for a given solution of the wave equation (f). Property 3 shows that there is an unique energy quantities (Σ^-, Σ^+) associated to the absolute value of a given solution of the wave equation ($|f|$). Finally, the last section deals with a special case of the wave equation in a dispersive medium with an example when the energy quantities are not equal. This work should be seen as an early beginning of a theory based on energy operators and new concepts may emerge from the redefinition of the wave equations in the future developments.

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