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On Congruences and BE-Relations in BE-Algebras

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Abstract

In this paper, a construction of a congruence having a given filter is presented. Also as a generalization of an BE-algebra homomorphism, the notion of a relation on BE-algebra, called an BE-relation is introduced and some fundamental properties to BE-algebras are discussed.

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1 Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Q. P. Hu and X. Li introduced a wide class of abstracts: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [5], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a dualization of a generation of a BCK-algebras. In this paper, a construction of a congruence having a given filter is presented. Also as a generalization of an BE-algebra homomorphism, the notion of a relation on BE-algebra, called an BE-relation is introduced and some fundamental properties to BE-algebras are discussed.

2 Preliminaries

In what follows, let X denote an BE-algebra unless otherwise specified.

By an *BE-algebra* we mean an algebra $(X; *, 1)$ of type (2, 0) with a single binary operation "$*$" that satisfies the following identities: for any $x, y, z \in X$,

(BE1) $x * x = 1$ for all $x \in X$,

(BE2) $x * 1 = 1$ for all $x \in X$,

(BE3) $1 * x = x$ for all $x \in X$,

(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

We introduce a relation "$\leq$" on X by $x \leq y$ imply $x * y = 1$. An BE-algebra $(X, *, 1)$ is said to be *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A non-empty subset S of an BE-algebra X is said to be a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$.

In an BE-algebra, the following identities are true:

(p1) $x * (y * x) = 1$.

(p2) $x * ((x * y) * y)) = 1$.

Definition 2.1. Let $(X, *, 1)$ be an BE-algebra and F a non-empty subset of X. Then F is said to be a *filter* of X if

(F1) $1 \in F$,

(F2) If $x \in F$ and $x * y \in F$, then $y \in F$.

Example 2.2. Let $X = \{1, a, b, c, d\}$ in which "$*$" is defined by

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

It is easy to know that X is a BE-algebra, and $F_1 = \{1, a\}, F_2 = \{1, b\}, F_3 = \{1, c\}, F_4 = \{1, a, b\}$ are filters of X.

3 Congruences

For a non-empty subset I of X we define the binary relation $\underset{I}{\sim}$ in the following way:

$$x \underset{I}{\sim} y \text{ if and only if } x * y \in I \text{ and } y * x \in I.$$

The set $\{b \mid a \underset{I}{\sim} b\}$ will be denoted by $[a]_I$.

Lemma 3.1. *In the above relation* $\underset{I}{\sim}$, *if* $1 \in I$, *then* $[1]_I = I$.

Proof. If $x \in I$, then $1 * x = x \in I$ and $x * 1 = 1 \in I$, which gives $1 \underset{I}{\sim} x$. Hence $I \subseteq [1]_I$.

Conversely, if $x \in [1]_I$, then $1 \underset{I}{\sim} x$, i.e. $x = 1 * x \in I$ by the definition of $\underset{I}{\sim}$. Thus $[1]_I \subseteq I$. Therefore $[1]_I = I$. □

Corollary 3.2. *If the above relation* $\underset{I}{\sim}$ *is reflexive, then* $[1]_I = I$.

Proof. Since $\underset{I}{\sim}$ is reflexive, we have $1 = x * x \in I$ for any $x \in X$ □

Corollary 3.3. *If* $\underset{I}{\sim}$ *is an equivalence relation, then*

$$x \in I, x * y \in I \text{ and } y \leq x \text{ imply } y \in I.$$

Proof. $1 \in I$ by Corollary 3.2. If $y \leq x$, then $y * x = 1 \in I$ which, together with $x * y \in I$ gives $x \underset{I}{\sim} y$. Since $\underset{I}{\sim}$ is an equivalence, x, y belong to the same class of $\underset{I}{\sim}$. By Corollary 3.2, $x \in I = [1]_I$ and in the consequence $y \in I$. □

Theorem 3.4. *Let X be a self-distributive BE-algebra and I a filter of X. Then* $\underset{I}{\sim}$ *is an equivalent relation on* X.

Proof. By Definition 2.1, it follows that $1 \in I$ since I is a filter of X. For any $x \in X$, we have $x * x = 1 \in I$, so $x \underset{I}{\sim} x$. If $x \underset{I}{\sim} y$, then by the definition of $\underset{I}{\sim}$, it is obvious that $y \underset{I}{\sim} x$. Now let $x \underset{I}{\sim} y$ and $y \underset{I}{\sim} z$. Then $x * y, y * x \in I$ and $y * z, z * y \in I$. Thus we get

$$\begin{aligned}(y * z) * ((x * y) * (x * z)) &= (y * z) * (x * (y * z)) \\ &= x * ((y * z) * (y * z)) \\ &= x * 1 = 1 \in I,\end{aligned}$$

which implies $(x * y) * (x * z) \in I$, and so $x * z \in I$ by (F2). Similarly, we have $z * x \in I$. Thus $\underset{I}{\sim}$ is an equivalence. □

Let $\underset{I}{\sim}$ be a binary relation on a set X. $\underset{I}{\sim}$ is said to be *compatible* if $a \underset{I}{\sim} b$ and $c \underset{I}{\sim} d$ imply $a * c \underset{I}{\sim} b * d$. A compatible equivalence on X is called a *congruence* on X.

Theorem 3.5. *The above relation $\underset{I}{\sim}$ in Theorem 3.4 is a congruence on X.*

Proof. If $x \underset{I}{\sim} u$ and $y \underset{I}{\sim} v$, then we have $(x * u) * ((u * y) * (x * y)) = (x*u)*((x*((u*y)*y))) = x*(u*((u*y)*y)) = x*((u*y)*(u*y)) = x*1 = 1 \in I$ and $x*u \in I$, which gives $(u*y)*(x*y) \in I$. Similarly, $(x*y)*(u*y) \in I$. Hence $x*y \underset{I}{\sim} u*y$. On the other hand, $(y*v)*((u*y)*(u*v)) = (y*v)*(u*(y*v)) = u*((y*v)*(y*v)) = u*1 = 1 \in I$ and $y*v \in I$ imply $(u*y)*(u*v) \in I$. In the same way, from $(v*y)*((u*v)*(u*y)) = 1 \in I$ and $v*y \in I$, we obtain $(u*v)*(u*y) \in I$. Thus $u*y \underset{I}{\sim} u*v$. Since the relation $\underset{I}{\sim}$ is transitive, we have $x*y \underset{I}{\sim} u*v$ which prove that the relation $\underset{I}{\sim}$ is a congruence. □

Lemma 3.6. *If the relation $\underset{I}{\sim}$ is a congruence on X, then*

$$[1]_I = \{x \in X \mid x \underset{I}{\sim} 1\}$$

is a filter of X.

Proof. Obviously, $1 \in [1]_I = \{x \in X \mid x \underset{I}{\sim} 1\}$. If $x, x*y \in [1]_I$, then $x*y \underset{I}{\sim} 1$ and $x \underset{I}{\sim} 1$. On the other hand, $y \underset{I}{\sim} y$ imply $1 \underset{I}{\sim} x*y \underset{I}{\sim} 1*y = y$. Thus $y \underset{I}{\sim} 1$, that is, $y \in [1]_I$. This completes the proof. □

Since $[1]_I = I$ for any congruence defined by Theorem 3.5, as a consequence of the above results, we obtain

Corollary 3.7. *Any filter of an BE-algebra is determined by some congruence.*

Corollary 3.8. *The lattice of all congruences of an BE-algebra is complete. The least congruence is defined by the filter $\{1\}$, the greatest by $I = X$.*

Definition 3.9. Let $X = (X, *, 1)$ and $X = (X', *', 1')$ be two BE-algebras. A mapping $f : X \to X'$ is called *an BE-algebra homomorphism* from X into X' if for any $x, y \in X$, $f(x*y) = f(x) *' f(y)$ holds. If in addition, $f(X) = X'$, then f is called an *epimorphism* and X' is said to be a *homomorphism image* of X.

4 BE-relation

We introduce the notion of a relation on BE-algebras, called *BE-relation*, which is a generalization of an BE-algebra homomorphism.

Definition 4.1. Let X and Y be BE-algebras. A non-empty relation $\mathcal{B} \subseteq X \times Y$ is called an *BE-relation* if it satisfies
$(\mathcal{B}1)(\forall x \in X)(\exists y \in Y)(x\mathcal{B}y)$,
$(\mathcal{B}2)(\forall x, y \in X)(\forall a, b \in Y)(x\mathcal{B}a, y\mathcal{B}b \Rightarrow (x * y)\mathcal{B}(a * b))$.

We usually denote such relation by $\mathcal{B} : X \to Y$. It is clear from $(\mathcal{B}1)$ and $(\mathcal{B}2)$that $1_X\mathcal{B}1_Y$.

Example 4.2. Let $X = \{1, a, b, c\}$ in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then X is an BE-algebra. Define a relation $\mathcal{B} : X \to X$ as follows: $1\mathcal{B}1, 1\mathcal{B}a, 1\mathcal{B}b, 1\mathcal{B}c, a\mathcal{B}1, a\mathcal{B}a$,
$a\mathcal{B}b, a\mathcal{B}c, b\mathcal{B}1, b\mathcal{B}a, c\mathcal{B}1, c\mathcal{B}a, c\mathcal{B}b$. It is easy to verify that $\mathcal{B}$ is an BE-relation.

Proposition 4.3. *Every homomorphism of BE-algebras is an BE-relation.*

Proof. Suppose that $\mathcal{B} : X \to Y$ be a homomorphism of BE-algebras. Clearly, $\mathcal{B}$ satisfies conditions $(\mathcal{B}1)$ and $(\mathcal{B}2)$. □

Note that every diagonal $\mathcal{B}$-relation on an BE-algebra (i.e., an $\mathcal{B}$ satisfying $x\mathcal{B}x$ for all $x \in X$ in which $x\mathcal{B}y$ is false whenever $x \neq y$) is clearly a BE-algebra homomorphism. But in general, the converse of Proposition 4.3 need not be true as seen in the following example.

Example 4.4. In Example 4.2, the BE-relation is not an BE-algebra homomorphism.

Let $\mathcal{B} : X \to Y$ be an BE-relation. For any $x \in X$ and $y \in Y$,

$$\mathcal{B}[x] := \{y \in Y \mid x\mathcal{B}y\}, \text{ and } \mathcal{B}^{-1}[y] := \{x \in X \mid x\mathcal{B}y\}.$$

Note that $\mathcal{B}[x]$ and $\mathcal{B}^{-1}[y]$ are not subalgebras of Y and X, respectively, as seen in the following example.

Example 4.5. Let $X = \{1, a, b\}$ in which "$*$" is defined by

$*$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then X is an BE-algebra. Define a relation $\mathcal{B} : X \to X$ as follows: $1\mathcal{B}1, a\mathcal{B}a, b\mathcal{B}b$. It is easy to verify that $\mathcal{B}$ is an BE-relation. Then $\mathcal{B}^{-1}[b] = \{b\}$ (resp. $\mathcal{B}[a] = \{a\}$) is not a subalgebra of X (resp. Y).

Theorem 4.6. *For any $\mathcal{B}$-relation $\mathcal{B} : X \to Y$, we have*

(1) $\mathcal{B}[1_X]$, called the zero image of $\mathcal{B}$, is a subalgebra of Y.

(2) $\mathcal{B}^{-1}[1_Y]$, called the kernel of $\mathcal{B}$ and denoted by $Ker\mathcal{B}$, is a subalgebra of X.

Proof. (1) Since $1_X\mathcal{B}1_Y$, we have $\mathcal{B}[1_X] \neq 0$. Let $y_1, y_2 \in \mathcal{B}[1_X]$. Then $1_X\mathcal{B}y_1$ and $1_X\mathcal{B}y_2$. It follows from ($\mathcal{B}2$) that $1_X\mathcal{B}(y_1 * y_2)$, that is, $y_1 * y_2 \in \mathcal{B}[1_X]$.

(2) Let $x_1, x_2 \in Ker\mathcal{B}$. Then we have $x_1\mathcal{B}1_Y$ and $x_2\mathcal{B}1_Y$. By using ($\mathcal{B}2$), we get $(x_1 * x_2)\mathcal{B}1_Y$, and so $x_1 * x_2 \in Ker\mathcal{B}$. This completes the proof. □

Proposition 4.7. *Let $\mathcal{B} : X \to Y$ be an BE-relation.Then we have*

(1) If $\mathcal{B}[a] \cap \mathcal{B}[b] \neq \emptyset$ where $a, b \in X$, then $a * b \in Ker\mathcal{B}$.

(2) If $\mathcal{B}^{-1}[u] \cap \mathcal{B}^{-1}[v] \neq \emptyset$ where $u, v \in Y$, then $u * v \in Ker\mathcal{B}[1_X]$.

Proof. (1) Let $a, b \in X$ be such that $\mathcal{B}[a] \cap \mathcal{B}[b] \neq \emptyset$. Taking $y \in \mathcal{B}[a] \cap \mathcal{B}[b]$, we have $a\mathcal{B}y$ and $b\mathcal{B}y$. It follows from ($\mathcal{B}2$) that $(a * b)\mathcal{B}(y * y) = (a * b)\mathcal{B}(y * y) = (a * b)\mathcal{B}1_Y$ so that $a * b \in Ker\mathcal{B}$.

(2) Let $x \in \mathcal{B}^{-1}[u] \cap \mathcal{B}^{-1}[v]$. Then $x\mathcal{B}u$ and $x\mathcal{B}v$. Using ($\mathcal{B}2$), we obtain $(x * x)\mathcal{B}(u * v) = 1_X\mathcal{B}(u * v)$, i.e., $u * v \in \mathcal{B}[1_X]$. This completes the proof. □

Theorem 4.8. *Let $\mathcal{B} : X \to Y$ be an BE-relation and let S be a subalgebra of X. Then*

$$\mathcal{B}[S] := \{y \mid x\mathcal{B}y \text{ for some } x \in S\}$$

is a subalgebra of Y.

Proof. Clearly, $\mathcal{B}[S] \neq \emptyset$ since $1_X\mathcal{B}1_Y$. Let $y_1, y_2 \in \mathcal{B}[S]$. Then $x_1\mathcal{B}y_1$ and $x_2\mathcal{B}y_2$ for some $x_1, x_2 \in S$. Using ($\mathcal{B}2$), we obtain $(x_1 * x_2)\mathcal{B}(y_1 * y_2)$ which implies that $y_1 * y_2 \in \mathcal{B}[S]$ since $x_1 * x_2$. Therefore $\mathcal{B}[S]$ is a subalgebra of Y. This completes the proof. □

Corollary 4.9. *Let $\mathcal{B} : X \to Y$ be an BE-relation. Then we have*

(1) $\mathcal{B}[X]$ is a subalgebra of Y,

(2) $\mathcal{B}[X] = \bigcup_{x\in X} \mathcal{B}[x]$,

(3) The zero image of $\mathcal{B}$ *is a subalgebra of* $\mathcal{B}[X]$.

Proof. (1) and (2) are straightforward. (3) Let $a, b \in \mathcal{B}[1_X]$. Then $1_X\mathcal{B}a$ and $1_X\mathcal{B}b$, and so $1_X\mathcal{B}(a * b)$, i.e., $a * b \in B[1_X]$. Therefore $\mathcal{B}[1_X]$ is a subalgebra of $\mathcal{B}[X]$. □

Theorem 4.10. *Let* $\mathcal{B} : X \to Y$ *be an* BE*-relation and let* T *be a subalgebra of* Y*. Then*

$$\mathcal{B}^{-1}[T] := \{x \in X \mid x\mathcal{B}y \textit{ for some } y \in T\}$$

is a subalgebra of X*.*

Proof. Obviously, $\mathcal{B}^{-1}[T] \neq \emptyset$ since $1_X\mathcal{B}1_Y$. Let $x_1, x_2 \in \mathcal{B}^{-1}[T]$. Then there exist $y_1, y_2 \in T$ such that $x_1\mathcal{B}y_1$ and $x_2\mathcal{B}y_2$. Note that $y_1 * y_2 \in T$ since T is a subalgebra of Y. It follows from ($\mathcal{B}$2) that $(x_1 * x_2)\mathcal{B}(y_1 * y_2)$ so that $x_1 * x_2 \in \mathcal{B}^{-1}[T]$. Hence $\mathcal{B}^{-1}[T]$ is a subalgebra of X. □

Corollary 4.11. *Let* $\mathcal{B} : X \to Y$ *be an* BE*-relation. Then*

(1) $\mathcal{B}^{-1}[Y]$ *is a subalgebra of* X,

(2) $\mathcal{B}^{-1}[Y] = \bigcup_{y\in Y} \mathcal{B}[y]$,

(3) The kernel of $\mathcal{B}$ *is a subalgebra of* $\mathcal{B}^{-1}[Y]$.

Proof. (1) and (2) are straightforward. (3) Let $x, y \in Ker\mathcal{B}$. Then $x\mathcal{B}1_Y$ and $y\mathcal{B}1_Y$. It follows from ($\mathcal{B}$2) that

$$(x * y)\mathcal{B}(1_Y * 1_Y) = (x * y)\mathcal{B}1_Y$$

so that $x * y \in Ker\mathcal{B}$. Bence $Ker\mathcal{B}$ is a subalgebra of $\mathcal{B}^{-1}[Y]$. This completes the proof. □

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