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Regular Elements of Some Semigroups of Order-Preserving Partial Transformations

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Abstract

Let X be a chain, $OP(X)$ the order-preserving partial transformation semigroup on X and $OI(X)$ the order-preserving 1–1 partial transformation semigroup on X. It is known that both $OP(X)$ and $OI(X)$ are regular semigroups. We extend these results by characterizing the regular elements of the semigroups $OP(X,Y), OI(X,Y), \overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ where $\emptyset \neq Y \subseteq X, OP(X,Y) = \{\alpha \in OP(X) \mid \operatorname{ran}\alpha \subseteq Y\}$, $\overline{OP}(X,Y) = \{\alpha \in OP(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}$, $OI(X,Y)$ and $\overline{OI}(X,Y)$ are defined analogously. The semigroups $OP(X,Y)$ and $\overline{OP}(X,Y)[OI(X,Y), \overline{OI}(X,Y)]$ may be counted as generalizations of $OP(X)[OI(X)]$. In addition, it is shown that each of these semigroups becomes a regular semigroup only the case that $Y = X$.

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1 Introduction

An element x of a semigroup S is called *regular* if $x = xyx$ for some $y \in S$ and S is called a *regular semigroup* if every element of S is regular. The set of all regular elements of S is denoted by $\operatorname{Reg}(S)$.

For a nonempty set X, let $T(X), P(X)$ and $I(X)$ be the full transformation semigroup on X, the partial transformation semigroup on X and the 1-1 partial transformation semigroup (the symmetric inverse semigroup) on X, respectively. The domain and the range (image) of $\alpha \in P(X)$ are denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. For $\alpha \in P(X)$ and $x \in \operatorname{dom}\alpha$, the image of x under α is written as $x\alpha$. Note that $T(X) = \{\alpha \in P(X) \mid \operatorname{dom}\alpha = X\}$,

$I(X) = \{\alpha \in P(X) \mid \alpha \text{ is 1-1}\}$ and $T(X)$ and $I(X)$ are subsemigroups of $P(X)$. Recall that for $\alpha, \beta \in P(X)$,

$$\begin{gathered}\operatorname{dom}(\alpha\beta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \subseteq \operatorname{dom}\alpha, \\ \operatorname{ran}(\alpha\beta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta \subseteq \operatorname{ran}\beta, \\ \text{for } x \in X,\ x \in \operatorname{dom}(\alpha\beta) \Leftrightarrow x \in \operatorname{dom}\alpha \text{ and } x\alpha \in \operatorname{dom}\beta.\end{gathered}$$

It is well-known that $T(X), P(X)$ and $I(X)$ are regular semigroups ([1], p.4). In fact, $I(X)$ is an inverse semigroup, i.e., for every $\alpha \in I(X)$, there is a unique $\beta \in I(X)$ such that $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$.

For a poset X and $\alpha \in P(X)$, α is said to be *order-preserving* if for $x, x^{'} \in \operatorname{dom}\alpha$, $x \leq x'$ implies $x\alpha \leq x'\alpha$. Let

$$OP(X) = \{\alpha \in P(X) \mid \alpha \text{ is order-preserving}\}.$$

We define $OT(X)$ and $OI(X)$ analogously. Kemprasit and Changphas [3] studied the regularity of $OT(X)$ for certain chains X. In [5], the authors characterized the regular elements of $OT(X)$ for any chain X and then showed that the results on $OT(X)$ provided in [3] can be obtained as its consequences.

Symmon [6] and Magill [4] have studied the subsemigroups

$$T(X,Y) = \{\alpha \in T(X) \mid \operatorname{ran}\alpha \subseteq Y\} \text{ and } \overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}$$

of $T(X)$, respectively where Y is a nonempty subset of a set X. Since $T(X,X) = \overline{T}(X,X) = T(X)$, we can count these semigroups as generalizations of $T(X)$. For a chain X, $OT(X,Y)$ is defined in [5] analogously, i.e., $OT(X,Y) = \{\alpha \in OT(X) \mid \operatorname{ran}\alpha \subseteq Y\}$. Then $OT(X,Y)$ may be considered as a generalization of $OT(X)$. In [5], the regular elements of $OT(X,Y)$ were characterized. The regularity of $OT(X,Y)$ was also determined. It was shown in [3] that for any chain X, $OP(X)$ and $OI(X)$ are regular semigroups. In this paper, the semigroups $OP(X,Y), OI(X,Y), \overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ are defined similarly where $\emptyset \neq Y \subseteq X$, i.e.,

$$\begin{aligned}OP(X,Y) &= \{\alpha \in OP(X) \mid \operatorname{ran}\alpha \subseteq Y\}, \\ OI(X,Y) &= \{\alpha \in OI(X) \mid \operatorname{ran}\alpha \subseteq Y\}, \\ \overline{OP}(X,Y) &= \{\alpha \in OP(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}, \\ \overline{OI}(X,Y) &= \{\alpha \in OI(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}.\end{aligned}$$

Then $OP(X,X) = \overline{OP}(X,X) = OP(X)$ and $OI(X,X) = \overline{OI}(X,X) = OI(X)$. Hence $OP(X,Y)$ and $\overline{OP}(X,Y)$ generalize $OP(X)$ while $OI(X,Y)$ and $\overline{OI}(X,Y)$ generalize $OI(X)$. Notice that $OP(X,Y) \subseteq \overline{OP}(X,Y)$ and $OI(X,Y) \subseteq \overline{OI}(X,Y)$. We characterize the regular elements of the semigroups $OP(X,Y), OI(X,Y), \overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ and show that $Y = X$ is necessary for each of these semigroups to be a regular semigroup.

2 The Semigroups $OP(X,Y)$ and $OI(X,Y)$

In this section, $\mathrm{Reg}(OP(X,Y))$, $\mathrm{Reg}(OI(X,Y))$ and the regularity of $OP(X,Y)$ and $OI(X,Y)$ are determined.

The following result is clearly seen. It will be often used in what follows:

Lemma 2.1. *Let X be a chain. If $\alpha \in OP(X)$ and $a, b \in \operatorname{ran}\alpha$ are such that $a < b$, then $s < t$ for all $s \in a\alpha^{-1}$ and $t \in b\alpha^{-1}$.*

Theorem 2.2. *Let X be a chain and $\emptyset \neq Y \subseteq X$. Then for $\alpha \in OP(X,Y)$, $\alpha \in \mathrm{Reg}(OP(X,Y))$ if and only if $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$.*

Proof. Assume that $\alpha \in \mathrm{Reg}(OP(X,Y))$. Let $\beta \in OP(X,Y)$ be such that $\alpha = \alpha\beta\alpha$. Then $\operatorname{ran}(\alpha\beta) \subseteq Y$, so

$$\operatorname{ran}\alpha = \operatorname{ran}(\alpha\beta\alpha) = (\operatorname{ran}(\alpha\beta) \cap \operatorname{dom}\alpha)\alpha \subseteq (Y \cap \operatorname{dom}\alpha)\alpha \subseteq \operatorname{ran}\alpha$$

which implies that $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$.

Conversely, assume that $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$. Then $x\alpha^{-1} \cap Y \neq \emptyset$ for all $x \in \operatorname{ran}\alpha$. For each $x \in \operatorname{ran}\alpha$, choose $d_x \in x\alpha^{-1} \cap Y$. Then $d_x\alpha = x$ for all $x \in \operatorname{ran}\alpha$. Define $\beta : \operatorname{ran}\alpha \to Y$ by

$$\beta = \begin{pmatrix} x \\ d_x \end{pmatrix}_{x \in \operatorname{ran}\alpha}.$$

Since $\alpha \in OP(X)$, by Lemma 2.1, β is order-preserving. Then $\beta \in OP(X,Y)$. Since for $x \in \operatorname{dom}\alpha$, $x\alpha \in \operatorname{dom}\beta$ and $x\alpha\beta \in \operatorname{dom}\alpha$, it follows that $\operatorname{dom}(\alpha\beta\alpha) = \operatorname{dom}\alpha$. If $x \in \operatorname{dom}\alpha$, then $x\alpha\beta\alpha = (x\alpha)\beta\alpha = d_{x\alpha}\alpha = x\alpha$. Therefore $\alpha = \alpha\beta\alpha$, so $\alpha \in \mathrm{Reg}(OP(X,Y))$, as desired. □

Notice that β defined in the proof of Theorem 2.2 is 1-1. Then $\beta \in OI(X,Y)$.

Theorem 2.3. *Let X be a chain and $\emptyset \neq Y \subseteq X$. Then for $\alpha \in OI(X,Y)$, $\alpha \in \mathrm{Reg}(OI(X,Y))$ if and only if $\operatorname{dom}\alpha \subseteq Y$.*

Proof. Assume that $\alpha \in \mathrm{Reg}(OI(X,Y))$. Since $OI(X,Y)$ is clearly a subsemigroup of $OP(X,Y)$, it follows that $\alpha \in \mathrm{Reg}(OP(X,Y))$. By Theorem 2.2, $\operatorname{ran}\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$. Then $(\operatorname{dom}\alpha)\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$, so $\operatorname{dom}\alpha = \operatorname{dom}\alpha \cap Y$ since α is 1-1. Hence $\operatorname{dom}\alpha \subseteq Y$.

Conversely, assume that $\operatorname{dom}\alpha \subseteq Y$. Then $\operatorname{dom}\alpha = \operatorname{dom}\alpha \cap Y$, so $\operatorname{ran}\alpha = (\operatorname{dom}\alpha)\alpha = (\operatorname{dom}\alpha \cap Y)\alpha$. From the proof of Theorem 2.2, there is a $\beta \in OI(X,Y)$ such that $\alpha = \alpha\beta\alpha$. Hence $\alpha \in \mathrm{Reg}(OI(X,Y))$. □

Corollary 2.4. *Let X be a chain, $\emptyset \neq Y \subseteq X$ and let $OS(X,Y)$ be $OP(X,Y)$ or $OI(X,Y)$. Then $OS(X,Y)$ is a regular semigroup if and only if $Y = X$.*

Proof. Suppose that $Y \subsetneq X$. Let $a \in X \smallsetminus Y$ and $b \in Y$. Then $\alpha = \binom{a}{b} \in OI(X,Y) \subseteq OP(X,Y)$. But $\operatorname{dom}\alpha \cap Y = \emptyset, \operatorname{ran}\alpha = \{b\}$ and $\operatorname{dom}\alpha = \{a\} \nsubseteq Y$, so by Theorem 2.2 and Theorem 2.3, $\alpha \notin \operatorname{Reg}(OS(X,Y))$. If $Y = X$, then $OP(X,Y) = OP(X)$ and $OI(X,Y) = OI(X)$ and both $OP(X)$ and $OI(X)$ are regular semigroups. Hence the corollary is proved. □

3 The Semigroups $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$

The purpose of this section is to characterize the elements of $\operatorname{Reg}(\overline{OP}(X,Y))$ and $\operatorname{Reg}(\overline{OI}(X,Y))$. The regularity of $\overline{OP}(X,Y)$ and $\overline{OI}(X,Y)$ is also considered.

Theorem 3.1. *Let X be a chain and $\emptyset \neq Y \subseteq X$. Then for $\alpha \in \overline{OP}(X,Y)$, $\alpha \in \operatorname{Reg}(\overline{OP}(X,Y))$ if and only if $\operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$.*

Proof. Assume that $\alpha \in \operatorname{Reg}(\overline{OP}(X,Y))$. Since $(\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y$, we have that $(\operatorname{dom}\alpha \cap Y)\alpha \subseteq \operatorname{ran}\alpha \cap Y$. To show that $\operatorname{ran}\alpha \cap Y \subseteq (\operatorname{dom}\alpha \cap Y)\alpha$, let $\beta \in \overline{OP}(X,Y)$ be such that $\alpha = \alpha\beta\alpha$. Let $x \in \operatorname{ran}\alpha \cap Y$. Then $x = a\alpha$ for some $a \in \operatorname{dom}\alpha$. Thus $x = a\alpha = a\alpha\beta\alpha = x\beta\alpha$ which implies that $x \in \operatorname{dom}\beta$ and $x\beta \in \operatorname{dom}\alpha$. It follows that $x \in \operatorname{dom}\beta \cap Y$ and hence $x\beta \in (\operatorname{dom}\beta \cap Y)\beta \subseteq Y$. We then deduce that $x\beta \in \operatorname{dom}\alpha \cap Y$. Consequently, $x = x\beta\alpha \in (\operatorname{dom}\alpha \cap Y)\alpha$. This proves that $\operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$.

For the converse, assume that $\operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$. Then $x\alpha^{-1} \cap Y \neq \emptyset$ for all $x \in \operatorname{ran}\alpha \cap Y$. For each $x \in \operatorname{ran}\alpha \cap Y$, choose $d_x \in x\alpha^{-1} \cap Y$ and for each $x \in \operatorname{ran}\alpha \smallsetminus Y$, choose $e_x \in x\alpha^{-1}$. Then $d_x\alpha = x$ for all $x \in \operatorname{ran}\alpha \cap Y$ and $e_x\alpha = x$ for all $x \in \operatorname{ran}\alpha \smallsetminus Y$. Define $\beta : \operatorname{ran}\alpha \to \operatorname{dom}\alpha$ by

$$\beta = \begin{pmatrix} x & u \\ d_x & e_u \end{pmatrix}_{\substack{x \in \operatorname{ran}\alpha \cap Y \\ u \in \operatorname{ran}\alpha \smallsetminus Y}}.$$

Then $(\operatorname{dom}\beta \cap Y)\beta = (\operatorname{ran}\alpha \cap Y)\beta = \{d_x \mid x \in \operatorname{ran}\alpha \cap Y\} \subseteq Y$. Since $\alpha \in OP(X)$, it follows from Lemma 2.1 that β is order-preserving. Hence $\beta \in \overline{OP}(X,Y)$. Since for $x \in \operatorname{dom}\alpha, x\alpha \in \operatorname{dom}\beta$ and $x\alpha\beta \in \operatorname{dom}\alpha$, we deduces that $\operatorname{dom}\alpha = \operatorname{dom}(\alpha\beta\alpha)$. If $x \in \operatorname{dom}\alpha$, then

$$x\alpha\beta\alpha = \begin{cases} d_{x\alpha}\alpha = x\alpha & \text{if } x\alpha \in Y, \\ e_{x\alpha}\alpha = x\alpha & \text{if } x\alpha \notin Y, \end{cases}$$

so $\alpha = \alpha\beta\alpha$. Thus $\alpha \in \operatorname{Reg}(\overline{OP}(X,Y))$, as desired. □

It can be seen that β constructed in the proof of Theorem 3.1 is 1-1. Then $\beta \in \overline{OI}(X,Y)$.

Theorem 3.2. *Let X be a chain and $\emptyset \neq Y \subseteq X$. Then for $\alpha \in \overline{OI}(X,Y)$, $\alpha \in \mathrm{Reg}(\overline{OI}(X,Y))$ if and only if $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq Y$.*

Proof. Assume that $\alpha \in \mathrm{Reg}(\overline{OI}(X,Y))$. Since $\overline{OI}(X,Y)$ is a subsemigroup of $\overline{OP}(X,Y)$, we have that $\alpha \in \mathrm{Reg}(\overline{OP}(X,Y))$. By Theorem 3.1, $\operatorname{ran}\alpha \cap Y = (\operatorname{dom}\alpha \cap Y)\alpha$. Then $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} = (\operatorname{dom}\alpha \cap Y)\alpha\alpha^{-1}$. Since $\alpha\alpha^{-1}$ is the identity mapping on $\operatorname{dom}\alpha$, it follows that $(\operatorname{dom}\alpha \cap Y)\alpha\alpha^{-1} = \operatorname{dom}\alpha \cap Y$. Hence $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} = \operatorname{dom}\alpha \cap Y \subseteq Y$.

Conversely, assume that $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq Y$. But $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq \operatorname{dom}\alpha$, so $(\operatorname{ran}\alpha \cap Y)\alpha^{-1} \subseteq \operatorname{dom}\alpha \cap Y$. Thus $(\operatorname{ran}\alpha \cap Y)\alpha^{-1}\alpha \subseteq (\operatorname{dom}\alpha \cap Y)\alpha \subseteq \operatorname{ran}\alpha \cap Y$. Since $\alpha^{-1}\alpha$ is the identity mapping on $\operatorname{ran}\alpha$, we have that $(\operatorname{ran}\alpha \cap Y)\alpha^{-1}\alpha = \operatorname{ran}\alpha \cap Y$. This implies that $(\operatorname{dom}\alpha \cap Y)\alpha = \operatorname{ran}\alpha \cap Y$. From the proof of Theorem 3.1, $\alpha = \alpha\beta\alpha$ for some $\beta \in \overline{OI}(X,Y)$. Hence $\alpha \in \mathrm{Reg}(\overline{OI}(X,Y))$, as desired. □

Corollary 3.3. *Let X be a chain, $\emptyset \neq Y \subseteq X$ and let $\overline{OS}(X,Y)$ be $\overline{OP}(X,Y)$ or $\overline{OI}(X,Y)$. Then $\overline{OS}(X,Y)$ is a regular semigroup if and only if $Y = X$.*

Proof. Suppose that $Y \subsetneq X$. Let $a \in X \setminus Y$ and $b \in Y$. Then $\alpha = \binom{a}{b} \in \overline{OS}(X,Y)$. Since $\operatorname{dom}\alpha \cap Y = \emptyset, \operatorname{ran}\alpha \cap Y = \{b\}$ and $b\alpha^{-1} = a \notin Y$,by Theorem 3.1 and Theorem 3.2, $\alpha \notin \mathrm{Reg}(\overline{OS}(X,Y))$. If $Y = X$, then $\overline{OP}(X,Y) = OP(X)$ and $\overline{OI}(X,Y) = OI(X)$ and both $OP(X)$ and $OI(X)$ are regular semigroups. Therefore the corollary is proved. □

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