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On Operators for which ${T^*}^2T^2 = (T^*T)^2$

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Abstract

In this paper we introduce a class (Q) of operators acting on a Hilbert space H : for any $T \in (Q)$, ${T^*}^2T^2 = (T^*T)^2$. We investigate some basic properties of such operators. We show that a quasinormal operator is in (Q). We give a condition under which an operator in (Q) becomes quasinormal. Also we show that an operator in (Q) is a $\theta-$operator. We show that the class (Q) and the class $(2N)$ of $2-$Normal operators are independent.

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1 Introduction

Throughout this paper H is a Hilbert space and $L(H)$ is the algebra of all bounded linear operators acting on H. If $T \in L(H)$ then T^* is its adjoint and $T = A + iB$ is its Cartesian decomposition. $T \in L(H)$ is called normal(respectively quasinormal , θ-operator) if $TT^* = T^*T$ (respectively $TT^*T = T^*T^2$, T^*T commutes with $T^* + T$).

In section 2 of this paper we investigate some basic properties of operators in (Q). In section 3 we show that a quasinormal operator is in (Q).We give a condition under which an operator in (Q) becomes quasinormal. We show that an operator T in (Q) is a θ-operator Thus T is normaloid.

2 Main results

In this section we investigate some basic properties of operators in (Q).

Proposition 1 *If $T \in (Q)$then so are*

(i) kT for any real number k

(ii) any$S \in L(H)$ that is unitarily equivalent to T

(iii) the restriction T/M of T to any closed subspace M of H that reduces T.

Proof. (i) The proof is straightforward so it is omitted.

(ii) Let $S \in L(H)$ be unitarily equivalent to T then there is a unitary operator $U \in L(H)$ such that $S = U^*TU$ which implies that $S^* = U^*T^*U$. Thus

$$S^{*^2}S^2 = U^*T^*UU^*T^*UU^*TUU^*TU = U^*T^{*^2}T^2U$$

and

$$(S^*S)^2 = (U^*T^*UU^*TU)^2 = (U^*T^*TU)^2 = U^*T^*TUU^*T^*TU = U^*(T^*T)^2U.$$

Since $T^{*^2}T^2 = (T^*T)^2$ we have $S^{*^2}S^2 = (S^*S)^2$. Thus $S \in (Q)$.

(iii) By ([1], Theorem 3, p.158) we have

$$\begin{aligned}(T/M)^{*^2}(T/M)^2 &= (T^{*^2}/M)(T^2/M)\\ &= T^{*^2}T^2/M\\ &= (T^*T)^2/M\\ &= (T^*T/M)^2\\ &= ((T/M)^*(T/M))^2.\end{aligned}$$

Thus $T/M \in (Q)$. ■

The following example shows that if unitarily equivalence in proposition 2.1(ii) is replaced by similarity then the result is not necessarily true.

Example 2 *Consider the two operators* $T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ *and* $X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ *acting on the two-dimensional Hilbert space* $\mathbb{R}^2$ *then it is clear that* $T = T^*$ *which implies that* $T \in (Q)$. *Now* $X^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ *and by direct computations one can show that* $XTX^{-1} = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} = A$ *(say). Now and again by direct computations one can show that* $A^{*^2}A^2 = \begin{bmatrix} 13 & -33 \\ -33 & 85 \end{bmatrix}$ *while* $(A^*A)^2 = \begin{bmatrix} 10 & -42 \\ -42 & 178 \end{bmatrix}$. *Thus* A *is similar to* T *but* $A \notin (Q)$.

The following example shows that (Q) is not convex.

Example 3 *Consider the two operators* $T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ *and* $S = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ *acting on* $\mathbb{R}^2$ *then it is clear that* $S \in (Q)$ *and by direct calculations one can show that*

$(T^*T)^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = T^{*^2}T^2$. *Thus* $T \in (Q)$. *Now consider the operator* $\frac{1}{2}T + \frac{1}{2}S = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{bmatrix} = A$ *(say) then it is direct that* $(A^*A)^2 = \begin{bmatrix} \frac{5}{16} & \frac{9}{8} \\ \frac{9}{8} & \frac{17}{4} \end{bmatrix} \neq \begin{bmatrix} \frac{1}{16} & \frac{3}{8} \\ \frac{3}{8} & \frac{13}{4} \end{bmatrix} = A^{*^2}A^2$. *Thus* (Q) *is not convex.*

Remark 4 *If* $T \in (Q)$ *such that* $T^2 = 0$ *then it is not necessarily that* $T = 0$. *For a counter example consider the operator* $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *acting on* $\mathbb{R}^2$.

The following is a characterization of operators in (Q).

Proposition 5 $T = A + iB \in (Q)$ *if and only if*

$$A^2B^2 + B^2A^2 = (AB)^2 + (BA)^2 \tag{i}$$

and

$$ABA^2 + B^2AB = A^2BA + BAB^2 \tag{ii}$$

Proof. Since $T = A + iB$ then $T^* = A - iB$. Thus we have

$$(T^*T)^* = (A^2 + B^2)^2 - (AB - BA)^2 + i[(A^2 + B^2)(AB - BA) + (AB - BA)(A^2 + B^2)] \tag{iii}$$

$$T^{*^2}T^2 = (A^2 - B^2)^2 + (AB + BA)^2 + i\left[(A^2 - B^2)(AB + BA) - (AB + BA)(A^2 - B^2)\right]. \tag{iv}$$

Suppose first that $T \in (Q)$ then the left hand sides of (iii) and (iv) are equal. Thus the right hand sides are equal which implies that

$$(A^2 + B^2)^2 - (AB - BA)^2 = (A^2 - B^2)^2 + (AB + BA)^2 \tag{v}$$

and

$$(A^2 + B^2)(AB - BA) + (AB - BA)(A^2 + B^2) = (A^2 - B^2)(AB + BA) - (AB + BA)(A^2 - B^2) \tag{vi}$$

Now simplifying the last two equations we get (i) & (ii). Suppose now that (i)&(ii) are true then it is straight forward that the right hand sides of (iii)&(iv) are equal. Thus $T \in (Q)$. ■

Corollary 6 *If* $T \in (Q)$ *then so is* T^*.

Proof. Suppose that $T = A + iB$ then $T^* = A - iB$. Now replacing B by $-B$ in $(i)\&(ii)$ in proposition 2.2 , then the two equations remains true. Thus $T^* \in (Q)$. ∎

If $T \in L(H)$ then it is not necessary that $(TT^*)^2 = T^2T^{*^2}$ as shown by the following example.

Example 7 *Consider the operator* $T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ *acting on* $\mathbb{R}^2$ *then direct calculations show that* $(T^*T)^2 = \begin{bmatrix} 20 & 12 \\ 12 & 8 \end{bmatrix} \neq T^{*^2}T^2 = \begin{bmatrix} 16 & 12 \\ 12 & 7 \end{bmatrix}$. *Thus* $T \notin (Q)$. *Also by direct calculations one can show that* $(TT^*)^2 = \begin{bmatrix} 26 & 6 \\ 6 & 2 \end{bmatrix} \neq \begin{bmatrix} 25 & 3 \\ 3 & 1 \end{bmatrix} = T^2T^{*^2}$.

Proposition 8 *If* $T \in (Q)$ *then* $(TT^*)^2 = T^2T^{*^2}$.

Proof. Since $T \in (Q)$, $T^* \in (Q)$. Thus we have $((T^*)^*(T^*))^2 = (T^*)^{*^2}(T^*)^2$ which implies that $(TT^*)^2 = T^2T^{*^2}$. ∎

Proposition 9 *If* $T \in (Q)$ *and* T^{-1} *exists, then* $T^{-1} \in (Q)$

Proof.

$$
\begin{aligned}
((T^{-1})^*T^{-1})^2 &= (T^*)^{-1}T^{-1})^2 \\
&= ((TT^*)^{-1})^2 \\
&= ((TT^*)^2)^{-1} \\
&= \left(T^2T^{*^2}\right)^{-1} \quad \text{(from proposition 2.3)} \\
&= \left(T^{*^2}\right)^{-1}\left(T^2\right)^{-1} \\
&= \left(T^{-1}\right)^{*^2}(T^{-1})^2.
\end{aligned}
$$

Thus $T^{-1} \in (Q)$. ∎

Proposition 10 *If* $T \in (Q)$ *then so is* $T + \lambda$ *for every real* λ.

Proof. Suppose that $T + \lambda \notin (Q)$, then

$$(T+\lambda)^*(T+\lambda))^2 - (T+\lambda)^{*^2}(T+\lambda)^2 \neq 0$$

which implies after simplifying that

$$T^{*^2}T + T^*T^2 + \lambda T^*T - T^*TT^* - TT^*T - \lambda TT^* \neq 0 \qquad \text{(i)}$$

Multiplying (i) on the right by T we get

$$T^{*^2}T^2 + T^*T^3 + \lambda T^*T^2 - (T^*T)^2 - TT^*T^2 - \lambda TT^*T \neq 0. \tag{ii}$$

Using the fact that $T^{*^2}T^2 = (T^*T)^2$, equation (ii) above becomes

$$T^*T^3 + \lambda T^*T^2 - TT^*T^2 - \lambda TT^*T \neq 0. \tag{iii}$$

Multiplying (iii) on the left by T^* we get

$$T^{*^2}T^3 + \lambda T^{*^2}T^2 - T^*TT^*T^2 - \lambda(T^*T)^2 \neq 0,$$

which implies that

$$T^{*^2}T^3 - T^*TT^*T^2 \neq 0. \tag{iv}$$

Multiplying (iv) above on the left by T^* we get $T^{*^3}T^3 - T^{*^2}TT^*T^2 \neq 0$, which implies that $T^*T^{*^2}T^2T - T^*(T^*T)^2T \neq 0$. A contradiction. Thus $(T+\lambda) \in (Q)$. ■

Proposition 11 *The class* (Q) *is closed in the strong operator topology.*

Proof. Let $\{Q_n\}$ be a sequence of operators in (Q) that converges strongly to an operator Q in $L(H)$ i.e. $Q_n \xrightarrow{s} Q$ then $\|Q_nx - Qx\| \longrightarrow 0$ as $n \longrightarrow \infty$ for each $x \in H$. Thus $\|Q_n^*x - Q^*x\| = \|(Q_n - Q)^*x\| \leq \|(Q_n - Q)^*\|\,\|x\| = \|Q_n - Q\|\,\|x\| \longrightarrow 0$ as $n \longrightarrow \infty$. Thus $Q_n^* \xrightarrow{s} Q^*$. Since the product of operators is sequentially continuous in the strong operators topology, $Q_n^*Q_n \xrightarrow{s} Q^*Q$ which implies that $(Q_n^*Q_n)^2 \xrightarrow{s} (Q^*Q)^2$. Also and for the same reason we have $Q_n^{*2} \xrightarrow{s} Q^{*2}$ and $Q_n^2 \xrightarrow{s} Q^2$ which implies that $Q_n^{*^2}Q_n^2 \xrightarrow{s} Q^{*^2}Q^2$. Since $\{Q_n\}$ is a sequence of operators in (Q), $Q_n^{*^2}Q_n^2 = (Q_n^*Q_n)^2$ which implies that $(Q^*Q)^2 = Q^{*^2}Q^2$. Thus $Q \in (Q)$ which means that (Q) is strongly closed. ■

In section 3 of this paper we study the relation between the class (Q) and some other classes of operators in $L(H)$. First we study the relation between (Q) and the class of quasinormal operators.

Proposition 12 *If* $T \in L(H)$ *is quasinormal, then* $T \in (Q)$.

Proof. Since T is quasinormal, then $TT^*T = T^*T^2$. Multiplying the last equation on the left by T^* we get $(T^*T)^2 = T^2T^{*^2}$. Thus $T \in (Q)$. ■

The following is a condition under which an operator in (Q) becomes quasinormal.

Proposition 13 *If* $T \in (Q)$ *and* $(T^*T)^3 = T^{*^3}T^3$, *then* T *is quasinormal.*

Proof.

$$(T^*T - TT^*)^2 = (T^*T)^2 - T^*T^2T^* - TT^{*^2}T + (TT^*)^2 \tag{i}$$

Multiplying (i) above on the left by T^* and on the left by T we get

$$T^*(T^*T - TT^*)^2T = T^*(T^*T)^2T - T^{*^2}T^2T^*T - T^*TT^{*^2}T^2 + T^*(TT^*)^2T. \tag{ii}$$

Since $T \in (Q)$, $(T^*T)^2 = T^2T^{*^2}$. Substituting in (ii) above we get

$$T^*(T^*T - TT^*)^2T = T^{*^3}T^3 - (T^*T)^3 - (T^*T)^3 + (T^*T)^3.$$

Using the assumption that $(T^*T)^3 = T^{*^3}T^3$ and the fact that $(T^*T)^3 = T^*(TT^*)^2T$ we get $T^*(T^*T - TT^*)^2T = 0$ which implies that $T^*(T^*T - TT^*)(T^*T - TT^*)T = 0$. Thus we have ([1],p. 140) $[(T^*T - TT^*)T]^*[(T^*T - TT^*)T] = 0$. Thus $(T^*T - TT^*)T = 0$ which implies that T is quasinormal. ■

Proposition 14 *If $T \in (Q)$, then $T \in \theta$.*

Proof. Let $T \in (Q)$ and $T \notin \theta$. Then

$$T^*TT^* + T^*T^2 - T^{*^2}T - TT^*T \neq 0. \tag{i}$$

Multiplying (i) above on right by T we get

$$(T^*T)^2 + T^*T^3 - T^{*^2}T^2 - TT^*T^2 \neq 0. \tag{1}$$

Using the fact that $(T^*T)^2 = T^{*^2}T^2$ (ii) becomes

$$T^*T^3 - TT^*T^2 \neq 0 \tag{iii}$$

Multiplying (iii) above on left by T^{*^2} we get

$$T^*T^{*^2}T^2T - T^*T^*TT^*TT \neq 0,$$

which implies that

$$T^*T^{*^2}T^2T - T^*(T^*T)^2T \neq 0,$$

which is not true. Thus $T \in \theta$. ■

Definition 15 *Let $T \in L(H)$. Then The spectrum of T is $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not invertible$\}$. The spectral radius of T is $r(T) = \sup\{\lambda : \lambda \in \sigma(T)\}$. It is well-known that for any $T \in L(H)$, $r(T) \leq \|T\|$. If $r(T) = \|T\|$, then T is called normaloid.*

Proposition 16 *If T is a $\theta-$operator then T is normaloid.*

Proof. ([2]) ■

Corollary 17 *If $T \in (Q)$, then $r(T) = \|T\|$.*

Next we study the relation between the class (Q) and the class of $2-$Normal operators which was introduced by the author in [2]. $T \in L(H)$ is called $2-$Normal if $T^2T^* = T^*T^2$. The author gave several characterizations of $2-$Normal operators such as: T is $2-$Normal if and only if $T^2T^* = T^*T^2$: if and only if T^2 is normal.

The following is an example of a $2-$Normal operator which is not in (Q).

Example 18 *Consider the operator* $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *acting on* $\mathbb{R}^2$*, then it can be shown that* $T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *which implies that* $T^2T^* = T^*T^2$*. Thus T is $2-$Normal. Now by direct calculations one can show that* $(T^*T)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = {T^*}^2T^2$*. Thus $T \notin (Q)$.*

Next we show that there is an operator in (Q) which is not $2-$Normal. To do so we need the following proposition and example.

Proposition 19 *If $T \in L(H)$ is isometry, then $T \in (Q)$.*

Proof. Since T is isometric, $T^*T = I$ which implies that $(T^*T)^2 = I$ and ${T^*}^2T^2 = T^*T = I$. Thus $T \in (Q)$. ■

The converse of proposition 3.4 is not in general true as shown in the next example

Example 20 *Consider the operator* $T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ *acting on* $\mathbb{R}^2$*, then T is self adjointthus $T \in (Q)$. Now by direct calculations one can show that* $T^*T = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \neq I$ *. Thus T is not isometric.*

Remark 21 *It is proved in [3] (Proposition 3.1) that if $T \in L(H)$ is both isometric and $2-$Normal, then T is unitary. Thus there are isometric operators which are not $2-$Normal. Now an isometric non $2-$Normal operator serves as an operator in (Q) which is not $2-$Normal. From Proposition 3.4 and Remark 3.1 we conclude that (Q) and $(2N)$ are independent.*

Proposition 22 *If $T \in L(H)$ is both $2-$Normal and in (Q), then T is normal.*

Proof.

$$\begin{aligned}(TT^* - T^*T)^*(TT^* - T^*T) &= (TT^* - T^*T)(TT^* - T^*T) \\ &= (TT^*)^2 - TT^{*^2}T - T^*T^2T^* + (T^*T)^2 \\ &= (TT^*)^2 - T^{*^2}T^2 - T^2T^{*^2} + (T^*T)^2 \quad \text{(since } T \in (2N)) \\ &= (TT^*)^2 - T^2T^{*^2} \quad \text{(since } T \in (Q)) \\ &= 0 \quad \text{(from proposition 2.3).}\end{aligned}$$

Thus by ([1], p.140), $TT^* - T^*T = 0$ which implies that T is normal. ■

Corollary 23 *If $T \in (Q)$ and T^2 is normal, then T is normal.*

Proof. Since T^2 is normal then by ([3], proposition 1.6, p.192) $T \in (2N)$. ■

Corollary 24 *If both T and T^* in $L(H)$ are quasinormal operators then T is normal* .

Proof. Since T is quasinormal, $TT^*T = T^*T^2$. Since T^* is quqsinormal, $T^*TT^* = TT^{*^2}$ which implies that $TT^*T = T^2T^*$ which implies that $T^2T^* = T^*T^2$, the result follows from proposition 3.5. ■

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