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On Generalized Probabilistic Metric Spaces

Ioan Goleţ

Department of Mathematics, "Politehnica" University
300006 Timişoara, Romania
ioan.golet@mat.upt.ro

Abstract

An enlargement of the concept of the distance between three points was given in [3], where B. C. Dhage studied generalized metric spaces. In the present paper we study some properties of a class of generalized probabilistic metric space and give some examples of such spaces. Some relationships between deterministic and probabilistic versions are stated.

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1 Introduction

The theory of metric spaces is of a fundamental importance in mathematics, computer science, statistics etc. Many problems can be solved by finding of appropriate metrics to make the measurements. Many generalizations of metric spaces were given and many papers and books have been published in this area [8-9], [11]. The positive number expressing the distance between two points is replaced by a probabilistic distribution function (in the sense of probabilistic theory [11]), or by a fuzzy set (in the sense of fuzzy metric spaces theory [2]). The distance between two points was also extended to three or more points [3-6]. These subjects was developed in various directions by a close connection with another areas of the mathematics. Different applications in engineering science and economics were also given. In the present paper we study some properties of a class of generalized probabilistic metric spaces and we give some examples of such spaces. Some relationships between deterministic and probabilistic versions are stated.
Let $\mathbf{R}$ denotes the set of real numbers, $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}$ and $I = [0, 1]$ the closed unit interval. The following system of axioms was proposed in [3] by B. C. Dhage for a distance between three points.

Definition 1.1 *Let X be a non empty set. A generalized metric space is a pair (X, d), where d is a mapping from $X \times X \times X$ into $\mathbf{R}_+$, which satisfies the following conditions :*

(1) $d(x, y, z) = 0$ *if, and only if,* $x = y = z$;
(2) $d(x, y, z) = 0$ *if at least two of* x, y, z *are equal;*
(3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$, *for every* x, y, z *in* X;
(4) $d(x, y, z) \geq d(x, y, u) + d(x, u, z) + d(u, y, z)$,
for every x, y, z, u *in* X.

This system of axioms gives appropriate properties for a distance between three points which, in a geometrically case, measures the perimeter of the triangle having as vertices these three points.

2 Preliminary Notes

In [10] K. Menger proposed a probabilistic concept of distance by replacing the number $d(p, q)$, the distance between points p, q by a distribution function $F_{p,q}$. This idea led to a large development of probabilistic analysis [7], [11].
A mapping $F : \mathbf{R} \to I$ is called a distribution function if it is non decreasing, left-continuous with $\inf F = 0$ and $\sup F = 1$. D^+ denotes the set of all distribution functions for that $F(0) = 0$, which are named distance distribution functions. Let F, G be in D^+ , then we write $F \leq G$ if $F(t) \leq G(t)$, for all $t \in \mathbf{R}$. If $a \in \mathbf{R}_+$ then H_a will be the element of D^+, for which $H_a(t) = 0$ if $t \leq a$ and $H_a(t) = 1 if t > a$. It is obvious that $H_0 \geq F$, for all $F \in D^+$. The set D^+ will be endowed with the natural topology defined by the modified Lévy metric d_L [11], named the topology of weak convergence. For every $F, G \in D^+$ we have the following properties :
(5) $F(t) > 1 - t$ if, and only if, $d_L(F, H_0) < t$.
(6) If $F \geq G$ then $d_L(G, H_0) \leq d_L(F, H_0)$.
(7) The metric space (D^+, d_L) is compact, and hence complete.
A t-norm T_1 is a two place function $T_1 : I \times I \to I$, which is associative, commutative, non decreasing in each place and such that $T_1(a, 1) = a$, for all $a \in [0, 1]$. A triangle function τ_1 is a binary operation on D^+ which is commutative, associative and for which H_0 is the identity, that is, $\tau_1(F, H_{0)}) = F$, for every $F \in D^+$ [11]. Let T_1 be a t-norm and let τ_1 be a triangle function. In the following sections we will consider the following functions $T : [0, 1]^3 \to [0, 1]$ given by $T(a, b, c) = T_1(T_1(a, b), c)$ and $\tau : [D^+]^3 \to D^+$ given by $\tau(F, G, H) = \tau_1(\tau_1(F, G), H)$. We name T a th-norm and τ a th-function. They have appropriate properties for writing a triangle inequality in generalized probabilistic metric spaces.

3 Main Results

Definition 3.1 *A generalized probabilistic metric space is an ordered triple $(X, \mathcal{F}, \tau)$, where X is a non empty set, $\mathcal{F}$ is a function defined on $X \times X \times X$ with values into D_+, τ is a th-function and the following conditions are satisfied*

(8) $F_{x,y,z} = H_0$ *if, and only if,* $x = y = z$,
(9) $F_{x,y,z} = H_{x,z,y} = H_{y,z,x}$, *for every* x, y, z *in* X.
(10) $F_{x,y,z} \geq \tau(F_{x,y,u}, F_{x,u,z}, F_{u,y,z})$, *for every* x, y, z, u *in* X.

If the probabilistic triangle inequality (10), named also a tetrahedral inequality, is given by a th-norm T :
(11) $F_{x,y,z}(t) \geq T(F_{x,y,u}((t_1), F_{x,u,z}(t_2), F_{u,y,z}(t_3))$, for every $t_1, t_2, t_3 \in \mathbf{R}_+$ such that $t_1 + t_2 + t_3 = t$, then $(X, \mathcal{F}, T)$ is called a generalized Menger metric space.

Remark 3.2 *It is easy to check that, every generalized metric space (X, d) can be made, in a natural way, a generalized Menger metric space by setting $F_{x,y,z}(t) = H_0(t - d(x, y, z))(t)$, for every $x, y, z \in X$, $t \in \mathbf{R}_+$ and $T = Min$.*

Proposition 3.3 *If T is a left continuous th-norm and τ_T is the th-function defined by $\tau_T(F, G, H)(t) = \sup_{t_1+t_2+t_3<t} T(F(t_1), G(t_2), H(t_3))$, $t > 0$ then, $(X, \mathcal{F}, \tau_T)$ is a generalized probabilistic metric space if and only $(X, \mathcal{F}, T)$ is a generalized Menger metric space.*

Example 3.4 *Let (X, d) be a generalized metric space and G a distance distribution function distinct from H_0. If we set $\mathcal{F}(x, y, z)(t) = G(t/d(x, y, z))$, for all x, y, z in X, $t \in \mathbf{R}$ and $T = Min$ then, the triple (X, d, F) induces a generalized probabilistic metric space $(X, \mathcal{F}, T)$. We have made the convention that $G(t/0) = G(\infty) = 1$ for $t > 0$ and $G(0/0) = G(0) = 0$. $(X, \mathcal{F}, T)$ is called the simple generalized probabilistic metric space generated by (X, d) and G. If $a > 0$ and $G = H_a$ then $F_{x,y,z}(t) = H_{ad(x,y,z)}(t)$. So, (X, d, H_a) determines a generalized metric space (X, d_a) $(d_a(x, y, z) = ad(x, y, z))$ homothetic to (X, d). Thus, generalized metric spaces are special cases of generalized probabilistic metric spaces, and each probabilistic distribution function induces on a generalized metric space a generalized probabilistic metric.*

Example 3.5 *Let $(\Omega, \mathcal{K}, P)$ be a complete probability measure space, let $(L, \|.\|)$ be a separable Banach space and let $(L, \mathcal{B})$ be the measurable space under the σ-algebra $\mathcal{B}$ of Borel subsets of $(L, ||.||)$. We denote by X the linear space of all almost surely equal class of random variables defined on $(\Omega, \mathcal{K}, P)$ with values in $(L, \mathcal{B})$[1].*

For all $x, y, z \in X$, $t \in \mathcal{R}_+$, *we define the mapping* $\mathcal{F} : X^3 \to D^+$ *given by* $\mathcal{F}(x, y, z) = F_{x,y,z}(t)$, *where*

$$F_{x,y,z}(t) = P(\{\omega \in \Omega : ||x(\omega) - y(\omega)|| + ||x(\omega) - y(\omega)|| + ||y(\omega) - z(\omega)|| < t\})$$

The triple $(X, \mathcal{F}, T_m)$ *verifies the axioms of a generalized Menger metric space* $(T_m(a, b, c) = Max\{a + b + c - 2\})$.

Example 3.6 *Let* (X, d) *be a generalized metric space. We define the mapping* $\mathcal{F} : X^3 \to D^+$ *given by* $F_{x,y,z}(t) = H_0$ *if* $x = y = z$ *and* $F_{x,y,z}(t) = \frac{t}{t+d(x,y,z)}$ *otherwise. Then the triple* $(X, \mathcal{F}, T)$ *verifies the axioms of a generalized Menger metric space for the th-norm* $(T(a, b, c) = Min\{a, b, c\}$.

Definition 3.7 *A sequence* $\{x_n\}$ *of points in a generalized probabilistic metric space* $(X, \mathcal{F}, \tau)$ *is said to be convergent to a point* $x \in X$ *if for every* $t > 0$ *there exists* $n_0 \in \mathbf{N}$ *such that*

$$F_{x_n,x_m,x}(t) > 1 - t,$$

for all $n, m \geq n_0$.

Definition 3.8 *We say that a sequence* $\{x_n\}$ *of a generalized probabilistic metric space* $(X, \mathcal{F}, \tau)$ *is a* $\mathcal{F}$*-Cauchy sequence if, for every* $t > 0$ *there exists* $n_0 \in \mathbf{N}$ *such that*

$$F_{x_n,x_m,x_p}(t) > 1 - t,$$

for all $m, p > n \geq n_0$.

Definition 3.9 *A self mapping* f *of a generalized probabilistic metric space* $(X, \mathcal{F}, T)$ *is said to be continuous if* $fx_n \to fx$, *whenever* $x_n \to x$.

Proposition 3.10 *Let* $\{x_n\}$ *be a sequence of points in a generalized probabilistic metric space* $(X, \mathcal{F}, T)$ *under a continuous th-norm* T. *Then :*

(*a*) $x_n \to x$ *if, and only if,* $F_{x_n,x_m,x}(t) \to H_0(t)$, *for all* $t > 0$.
(*b*) $\{x_n\}$ *is a* $\mathcal{F}$*-Cauchy sequence if, and only if,* $F_{x_n,x_m,x_p}(t) \to H_0(t)$, *for all* $t > 0$.

The following theorem shows us that, under some th-norm, on a generalized probabilistic metric space can be defined a deterministic generalized metric and the topologies induced are equivalent.

Theorem 3.11 *Let* $(X, \mathcal{F}, T)$ *be a generalized Menger metric space under a continuous th-norm* T *such that* $T \geq T_m$ *and let consider the mapping* $d : X^3 \to \mathcal{R}$ *defined by*

$$d(x, y, z) = sup\{\varepsilon \in [0, 1) : F_{x,y,z}(\varepsilon) \geq 1 - \varepsilon\}.$$

Then we have :

(a) $d(x, y, z) < t$ *if and only if* $F_{x,y,z}(t) > 1 - t$.
(b) (X, d) *is a D-metric space.*
(c) *The convergence under the generalized probabilistic metric* $\mathcal{F}$ *is equivalent with convergence under the generalized metric* d.

Proof. (a) If $1 < t$, then $d(x, y, z) \geq 1 < t$ and also $F_{x,y,z}(t) \geq 0 > 1 - t$. Suppose $d(x, y, z) < t \geq 1$ and choose δ such that $d(x, y, z) < \delta < t \geq 1$. Then $F_{x,y,z}(t) \geq F_{x,y,z}(\delta) > 1 - \delta > 1 - t$.

Conversely, suppose $F_{x,y,z}(t) > 1 - t$, where $0 < t \geq 1$. Then there exists $0 < \delta_0 < t$ such that $d(x, y, z) < \delta_0 < t$. If $d(x, y, z) > \delta$ for each $\delta < t$ then $F_{x,y,z}(\delta) \geq 1 - \delta$ for each $\delta < t$. This implies that $F_{x,y,z}(t) = \lim_{\delta \to t^-} F_{x,y,z}(\delta) \geq \lim_{\delta \to t^-} (1 - \delta) = 1 - t$. So, we have a contradiction, hence there exists $0 < \delta_0 < t$ such that $d(x, y, z) < \delta_0 < t$.

(b) If $x = y = z$ then $F_{x,y,z} = H_0$ and $d(x, y, z) = sup\{0\} = 0$. Inverse implications are also true. So $d(x, y, z) = 0 \Leftrightarrow x = y = z$. The symmetry of $F_{x,y,z}$ is equivalent with that of $d(x, y, z)$.
Now, we show that d satisfies the tetrahedral inequality for a generalized metric. It is sufficient to show that $d(x, y, u) < \varepsilon_1$, $d(x, u, z) < \varepsilon_2$, $d(u, y, z) < \varepsilon_3$, implies $d(x, y, z) < \varepsilon$, where $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$. If $d(x, y, u) < \varepsilon_1$, $d(x, u, z) < \varepsilon_2$, $d(u, y, z) < \varepsilon_3$ then there exists $\delta_i, i = 1, 2, 3$ such that $d(x, y, u) < \delta_1 < \varepsilon_1$, $d(x, u, z) < \delta_2 < \varepsilon_2$, $d(u, y, z) < \delta_3 < \varepsilon_3$. By the inequality (12) we have :
$F_{x,y,z}(\varepsilon) \geq F_{x,y,z}(\delta_1 + \delta_2 + \delta_3) \geq T(F_{x,y,u}(\delta_1), F_{x,u,z}(\delta_2), F_{u,y,z}(\delta_3)) \geq$
$T_m(1 - \delta_1, 1 - \delta_2, 1 - \delta_3) \geq 1 - (\delta_1 + \delta_2 + \delta_3) > 1 - \varepsilon$.
By statement (a) it follows that $d(x, y, z) < \varepsilon$. Thus, the mapping d satisfies the tetrahedral inequality (4) for a generalized metric. The statement (c) follows as a consequence of (a) and (b).
In the sequel we show that a generalized probabilistic metric structure can be induced on a arbitrary set by a function defined on that set with values into a generalized probabilistic metric space.

Theorem 3.12 *Let* g *be a injective mapping defined on a generalized Menger metric space* $(X, \mathcal{F}, T)$ *into itself. Then the following statements are true :*

(a) *The mapping* $\mathcal{F}^g$ *defined on* $X \times X \times X$ *with values in* $\mathcal{D}^+$, *by* $\mathcal{F}^g(x, y, z) = F_{g(x),g(y),g(z)}$ *is a generalized probabilistic metric on* X, *that is,* $(X, \mathcal{F}^g, T)$ *is a generalized Menger metric space under the same t-norm* T.

(b) *If* $X_1 = g(X)$ *and* $(X_1, \mathcal{F}, T)$ *is a complete generalized Menger metric space then,* $(X, \mathcal{F}^g, T)$ *is also a complete generalized Menger metric space.*

(c) *If* $(X_1, \mathcal{F}, T)$ *is compact then* $(X, \mathcal{F}^g, T)$ *is also compact.*

Proof. We will prove only the statement (c). Let $(x_n)_{n\geq 1}$ be a sequence in X. Then $(u_n)_{n\geq 1}$ with $u_n = g(x_n)$ is a sequence in X_1, which is a compact Menger D-metric space. Now, we can find a subsequence $\{v_n : n \geq 1\} \subset \{u_n : n \geq 1\}$ convergent to an element $v \in X_1$. This is equivalent to $F_{v_n,v_m,v}(t) \to H_0(t)$,

$(n, m \to \infty)$, for every $t > 0$. If we set $y_n = g^{-1}(v_n)$, $y_m = g^{-1}(v_m)$ and $y = g^{-1}(v)$ then, we have :

$$F^g_{y_n,y_m,y}(t) = F_{g(y_n),g(y_m),g(y)}(t) = F_{v_n,v_m,v}(t) \to H_0(t), (n, m \to \infty),$$

for every $t > 0$. This show us that $(X, \mathcal{F}^g, T)$ is a compact generalized Menger metric space.

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