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M-Complementary Distance Uniform Matrix of a Graph

Beena Koshy [†] **and K. A. Germina** [‡]

Research Center & PG Department of Mathematics
Mary Matha Arts & Science College (Kannur University)
Mananthavady, Kerala, India
beenakoshy1@yahoo.co.in[†]
srgerminaka@gmail.com[‡]

Abstract

A graph $G = (V, E)$ is *Complementary Distance Pattern Uniform* if there exists $M \subset V(G)$ such that $f_M(u) = \{d(u, v) : v \in M\}$, for every $u \in V(G) - M$, is independent of the choice of $u \in V(G) - M$ and the set M is called the Complementary Distance Pattern Uniform Set (CDPU set). The M-Complementary Distance Uniform Matrix (M-CDU Matrix), $C_M(G) = [c_{ij}]$ of G of order n is a $p \times m$ matrix, where m is the cardinality of the CDPU set M and $p = |V(G) - M|$, having entries $c_{ij} = d(v_i, v_j)$ where $v_i \in V(G)$ and $v_j \in M$. In this paper, we initiate a study on M-CDU Matrix of a graph and a k-colorable CDPU graph.

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1 Introduction

For all terminology and notation in graph theory, not defined specifically in this paper, we refer the reader to F. Harary [5]. Unless mentioned otherwise, all the graphs considered in this note are simple, self-loop-free and finite.

Let $G = (V, E)$ represent the structure of a chemical molecule. Often, a topological index (TI), derived as an invariant of G, is used to represent a chemical property of the molecule. There are a number of TIs based on distance concepts in graphs [7] and some of them could be designed using distance patterns of vertices in a graph. There are strong indications in the literature [7] that the notion of CDPU sets in G could be used to design a class of TIs that represent certain stereochemical properties of the molecule.

B.D.Acharya [10] define the M - distance pattern of a vertex as follows :

Definition 1.1. *[10] Given an arbitrary non-empty subset M of vertices in a graph $G = (V, E)$, each vertex $u \in G$ is associated with the set $f_M(u) = \{d(u, v) : v \in M\}$, where $d(u, v)$ denotes the usual distance between the vertices u and v in G, is called the M- vertex distance pattern of u. If for a subset M of vertices in a graph $G = (V, E)$, f_M is injective, then the set M is called the distance pattern distinguishing set (DPD-set in short).*

Germina and Beena [2] defined *Complementary Distance Pattern Uniform (CDPU) Graph* as follows

Definition 1.2. *[2] If $f_M(u)$ is independent of the choice of $u \in V - M$, then G is called a Complementary Distance Pattern Uniform (CDPU) Graph. The set M is called the CDPU set. The least cardinality of CDPU set in G is called the* CDPU number *of G, denoted $\sigma(G)$.*

We need the following known results.

Theorem 1.3. *[2] Every connected graph has a CDPU set.*

Theorem 1.4. *[2] A graph G has $\sigma(G) = 1$ if and only if G has atleast one vertex of full degree.*

Theorem 1.5. *[2] For any integer n, $\sigma(P_n) = n - 2$.*

Corollary 1.6. *[2]* $\sigma(K_{a_1,a_2}) = 2.$

Theorem 1.7. *[2]* $\sigma(C_n) = \begin{cases} \frac{n}{2} & ;\ when \quad n \geq 8 \quad is \quad even \\ n-2 & ;\ when \quad n \quad is \quad odd \end{cases}$

2 M-Complementary Distance Uniform Matrix of CDPU graphs

We define M-Complementary Distance Uniform Matrix of CDPU graphs.

Definition 2.1. *The $M-$ Complementary Distance Uniform Matrix ($M-CDU$ Matrix) $C_M(G) = [c_{ij}]$ of G of order n is a $p \times m$ matrix, where m is the cardinality of the CDPU set M and $p = |V(G) - M|$, having entries $c_{ij} = d(v_i, v_j)$ where $v_i \in V(G)$ and $v_j \in M$.*

The following observations are immediate.

Observation 2.2. *Let G be a graph with CDPU set M. Then, all the entries in $M-$Complementary distance uniform matrix are greater than or equal to one and hence the matrix have no null rows.*

Observation 2.3. *Let G be a graph with CDPU set M. Then, the entries of each rows of the $M-CDU$ matrix are same and hence row-sum of every row is a constant.*

Observation 2.4. *Largest entry in $C_M(G)$ should be less than or equal to the diameter of the graph G.*

Remark 2.5. *Let G be a graph with CDPU set M. Then, the entries of each columns of the $M-CDU$ matrix need not be same. Hence, it is of interest to characterize those CDPU graphs having the entries of each columns of the $M-CDU$ matrix same.*

Observation 2.6. *The entries in each row is same as the set $f_M(u)$ of a CDPU graph for $u \in V - M$.*

The following Propositions are immediate from the above observations.

Proposition 1. *The rows in $C_M(G)$ are linearly dependent.*

Proposition 2. *If the entries of each row in $C_M(G)$ is of the form $1, 3, 5, \ldots$; the set of all consecutive odd integers, then M is an independent set.*

Proof. Since the entries of each row in $C_M(G)$ is of the form $1, 3, 5, \ldots$; $f_M(v) = \{1, 3, 5, \ldots\}$, for every $v \in V - M$. Let $M = \{u_1, u_2, \ldots\}$. Then for some $u \in V - M$, $d(u, u_1) = 1, d(u, u_2) = 3$ and so on.Hence it is clear that u_1 and u_2 are not adjacent. □

Theorem 2.7. *If a graph G contains exactly one full degree vertex, then the $M-CDU$ matrix is of the form*

$$\begin{bmatrix} 1 \\ 1 \\ \cdots \\ \cdots \\ 1 \end{bmatrix}$$

Proof. Since G contains exactly one full degree vertex, say, v, then $M = \{v\}$. Hence for $u_1, u_2, \ldots, u_k \in V - M$, $c_{11} = d(u_1, v) = 1$, $c_{21} = d(u_2, v) = 1$ and so on. □

Theorem 2.8. *If a graph G contains two full degree vertices, then the $M-CDU$ matrix is of two forms: one is of the form as described in Theorem 2.7 and the other is of the form*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \cdots & \cdots \\ \cdots & \cdots \\ 1 & 1 \end{bmatrix}$$

Proof. Since G contains two full degree vertices, say, u, v, then $M_1 = \{u\}, M_2 = \{v\}$ and $M_3 = \{u, v\}$ are CDPU sets. Then from Theorem 2.7, $C_{M_1}(G)$ and $C_{M_2}(G)$ are clear. Let $V - M = \{v_1, v_2, \ldots, v_m\}$. Hence
$c_{11} = d(v_1, u) = 1,\ c_{12} = d(v_1, v) = 1,$
$c_{21} = d(v_2, u) = 1,\ c_{22} = d(v_2, v) = 1,$
$\cdots\cdots\cdots\cdots$
$c_{m1} = d(v_m, u) = 1,\ c_{m2} = d(v_m, v) = 1.$ □

Lemma 2.9. *The $M-$ CDU matrix of P_n is given by*

$$\begin{bmatrix} 1 & 2 & \dots & \dots & n-2 \\ n-2 & n-3 & \dots & \dots & 1 \end{bmatrix}$$

Proof. For a path P_n with n vertices, the CDPU set is the set of all cut vertices with $|M| = n-2$. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Then $M = \{v_2, v_3, \dots, v_{n-1}\}$. Hence
$c_{11} = d(v_1, v_2) = 1,\ c_{12} = d(v_1, v_3) = 2,\ c_{13} = d(v_1, v_4) = 3, \dots, c_{1(n-2)} = d(v_1, v_{n-1}) = n-2$ and
$c_{21} = d(v_n, v_{n-1}) = 1,\ c_{22} = d(v_n, v_{n-2}) = 2,\ c_{23} = d(v_n, v_{n-3}) = 3, \dots, c_{2(n-2)} = d(v_n, v_1) = n-2$.
Hence the proof. □

Corollary 2.10. *For a path P_n, sum of each column is the same and is equal to $n-1$. Also the sum of each row is equal to $\frac{(n-2)(n-1)}{2}$.*

For a cycle C_n with n even, the set of all alternate vertices form the CDPU set for C_n. The following Illustration shows the M-CDU matrix of C_{10}.

Illustration 1. *Consider C_{10} with $V(C_{10}) = \{v_1, v_2, \dots, v_{10}\}$. Then $M = \{v_2, v_4, v_6, v_8, v_{10}\}$. The M-CDU matrix of C_{10} is given by*

$$\begin{bmatrix} 1 & 3 & 5 & 3 & 1 \\ 1 & 1 & 3 & 5 & 3 \\ 3 & 1 & 1 & 3 & 5 \\ 5 & 3 & 1 & 1 & 3 \\ 3 & 5 & 3 & 1 & 1 \end{bmatrix}$$

Corollary 2.11. *In cycles, the M-CDU matrix is an $\frac{n}{2} \times \frac{n}{2}$ matrix with diagonal elements one. Also sum of rows = sum of columns.*

Proposition 3. *Sum of the rows in the $M-$ CDU matrix of a cycle C_n with $n = 2d$ is given by* $\begin{cases} \frac{d^2+1}{2} & ;\ when\ \ d\ \ is\ \ odd \\ \frac{d^2}{2} & ;\ when\ \ d\ \ is\ \ even \end{cases}$

Proof. Let $n = 2d$.
Case 1 d is odd
Sum of the rows $= 2(d-1) + 2(d-3) + \dots + 2 \times 1$

$= 2[1 + 3 + \cdots + (d - 1)$
$= 2\frac{\frac{d}{2}}{2}(1 + d - 1)$
$= \frac{d}{2}d$
$= \frac{d^2}{2}.$

Case 2 d is even

Sum of rows $= d + 2(d - 2) + 2(d - 4) + \cdots + 2 \times 1$
$= d + 2[1 + 3 + \cdots + (d - 2)]$
$= d + 2\frac{d-1}{4}(d - 1)$
$= d + \frac{(d-1)^2}{2}$
$= \frac{d^2+1}{2}.$ □

Proposition 4. *Sum of the rows in the M− CDU matrix of a cycle C_n with n odd is given by $d^2 + d - 1$.*

Proof. Sum of the rows $= 1 + 4 + 6 + \cdots + 2d$
$= 1 + 2(2 + 3 + \cdots + d)$
$= 1 + 2 \times \frac{d-1}{2}(d + 2)$
$= 1 + (d - 1)(d + 2)$
$= d^2 + d - 1$ □

Theorem 2.12. *The M-CDU matrix of a complete bipartite graph $K_{m,n}$ corresponding to the M as one of the partite sets is*

$$\begin{bmatrix} 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \dots & 1 \end{bmatrix}$$

Proof. Let $G = K_{m,n}$ be a complete bipartite graph and let U, W, be the partition of $V(G)$, where $|U| = m$ and $|W| = n$. Let $U = \{u_1, u_2, \dots, u_m\}$ and $W = \{v_1, v_2, \dots, v_n\}$. Choose M as one of the partite sets. Let $M = U = \{u_1, u_2, \dots, u_m\}$. Then

$c_{11} = d(v_1, u_1) = 1,\ c_{12} = d(v_1, u_2) = 1, \dots, c_{1m} = d(v_1, u_m) = 1$ *and*
$c_{21} = d(v_2, u_1) = 1,\ c_{22} = d(v_2, u_2) = 1, \dots, c_{2m} = d(v_2, u_m) = 1$
$\dots\dots\dots\dots\dots\dots\dots$
$c_{n1} = d(v_n, u_1) = 1,\ c_{n2} = d(v_n, u_2) = 1, \dots, c_{nm} = d(v_n, u_m) = 1.$ □

Corollary 2.13. *The sum of the each row in the M-CDU matrix of a complete bipartite graph $K_{m,n}$ is equal to n and sum of each column is equal to m.*

3 k-Colorable CDPU graphs

Let M be a CDPU set of graph G. Consider the induced subgraph $< M >$ of G. $< M >$ may or may not be connected. An interesting question that arises is to determine the chromatic number of the induced subgraph $< M >$ of G. We recall the following existing results.

Definition 3.1. *[3] A k-coloring of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. A graph is k-colorable if it has a proper k-coloring. The chromatic number $\chi(G)$ is the least k such that G is k-colorable.*

Now, we define k-coloring of a CDPU-graph as follows

Definition 3.2. *A graph G such that $< M >$ can be colored using k colors is called a k-colorable CDPU graph. The chromatic number $\chi_M(G)$ of a CDPU graph is the least k such that $\langle M \rangle$ is k-colorable.*

Clearly, $\chi_M(G) \leq \chi(G)$ and $\chi_M(G) \leq |M|$.

We are now able to determine the k-coloring of some CDPU graphs.

Theorem 3.3. *For a cycle C_n,* $\chi_M(G) = \begin{cases} 1 & ; \text{ when } n \text{ is even} \\ 2 & ; \text{ when } n \text{ is odd} \end{cases}$

Proof. **Case 1**: n is even

Since n is even, the set of all alternate vertices of C_n form a CDPU set and $< M >= \overline{K}_{\frac{n}{2}}$, is disconnected and hence is 1-colorable.

Case 2: n is odd

In this case, the set of all adjacent $n-2$ vertices form a CDPU set for C_n and $< M >= P_{n-2}$, and hence 2-colorable. □

Theorem 3.4. *Any graph with a full degree vertex is 1-colorable CDPU.*

Proof. Let G be graph with a full degree vertex v. Then, $M = \{v\}$ is the CDPU set for G. Hence, $< M >= \overline{K}_1$, is 1-colorable. □

Theorem 3.5. $\chi_M(K_{m,n}) = 2$.

Proof. The proof follows from the fact that the induced graph of the CDPU-set M is nothing but a path on two vertices. □

Theorem 3.6. $\chi_M(P_n) = 1$ *if and only if* $n \leq 5$.

Proof. When $n = 2, 3$, P_2 and P_3 contains a vertex of full degree and the result follows from Theorem 3.4. Let $n = 4$, and let $V(P_4) = \{v_1, v_2, v_3, v_4\}$. Hence, $M = \{v_1, v_4\}$ and $< M >= \overline{K}_2$, which is clearly 1-colorable. Let $n = 5$, let $V(P_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Therefore, $M = \{v_1, v_3, v_5\}$, which is 1-colorable.

Conversely, let $\chi_M(P_n) = 1$. We show that $n \leq 5$. If possible assume $n \geq 6$. Then, M is the set of all cut vertices of P_n, which in turn implies $< M >= P_{n-2}$, and hence is 2-colorable. Hence $\chi_M(P_n) = 1$ if and only if $n \leq 5$. □

Corollary 3.7. $\chi_M(P_n) = \begin{cases} 1 & ; \quad if \quad n \leq 5 \\ 2 & ; \quad if \quad n \geq 6 \end{cases}$

Color Classes and CDPU Sets

Another way of viewing a proper k-coloring is an assignment of vertices to sets, called color classes, where each set represents vertices with the same color pattern. For the coloring to be proper, each color class must be an independent set of vertices. Our next target is to check whether the color classes of a graph G form a CDPU set for G.

Theorem 3.8. *The color classes of a path P_n form a CDPU set if and only if $n \leq 5$.*

Proof. We have $\chi(P_n) = 2$, for $n \geq 2$. So there are exactly two color classes for P_n.

Let $V(P_2) = \{v_1, v_2\}$. The color classes are $c_1 = \{v_1\}$ and $c_2 = \{v_2\}$. Both c_1 and c_2 are CDPU sets for G.

Let $V(P_3) = \{v_1, v_2, v_3\}$. Then, $c_3 = \{v_1, v_3\}$ and $c_4 = \{v_2\}$. Also both c_3 and

c_4 are CDPU sets for P_3.
Let $V(P_4) = \{v_1, v_2, v_3, v_4\}$. Then, $c_5 = \{v_1, v_3\}$ and $c_6 = \{v_2, v_4\}$ are CDPU sets for G.
Let $V(P_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Whence, $c_7 = \{v_1, v_3, v_5\}$ and $c_8 = \{v_2, v_4\}$ are CDPU sets for P_5.

Conversely, assume that the color classes of P_n form a CDPU set. We show that $n \leq 5$. If possible, assume that $n \geq 6$. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Then, the color classes are $c_1 = \{v_1, v_3, \ldots\}$ and $c_2 = \{v_2, v_4, \ldots\}$. Since by Theorem ? for $n \geq 6$, the set of all cut vertices is the only CDPU set for P_n, both c_1 and c_2 are not CDPU sets for P_n. Hence, we conclude that $n \leq 5$. □

Invoking Theorem 3.6 and Theorem 3.8, we have

Theorem 3.9. *The color classes of a path P_n form a CDPU set if and only $\chi_M(P_n) = 1$.*

Proposition 5. *The color classes of C_n, n even, are CDPU sets.*

Proof. Consider the case when n is even. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Since $\chi(C_n) = 2$, then C_n has two color classes; viz., $c_1 = \{v_1, v_3, \ldots\}$ and $c_2 = \{v_2, v_4, \ldots\}$. We have already proved that c_1 and c_2 are CDPU sets for C_n. □

Proposition 6. *The color classes of an odd cycle does not possess CDPU sets.*

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, n\}$, where $n \geq 5$ is odd. Since $\chi(C_n) = 3$, there are three color classes; viz. $c_1 = \{v_1, v_3, \ldots\}, c_2 = \{v_2, v_4, \ldots\}$ and $c_3 = \{v_{n-1}\}$. Invoking Theorem 1.7, c_1, c_2 and c_3 are not CDPU sets. □

Invoking Proposition 5 and Proposition 6, we have

Theorem 3.10. *The color classes of C_n are CDPU sets if and only if n is even and $n = 3$.*

Scope and Conclusion

The matrix are commonly used to represent graphs for computer processing. The advantage of representing graphs in matrix form lies on the fact that many results of matrix algebra can be readily applied to study the structural properties of graphs from an algebraic point of view. An algorithm to determine the subsets of vertex set that act as both color classes and CDPU sets are of very much interest for chemists. The determination of the subset of vertex set that are color classes are shown to be NP-complete. Hence the complexity of determining the CDPU sets are also expected to be NP-complete.

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