International Mathematical Forum, 5, 2010, no. 45, 2237 - 2247

(k, r)-Arithmetic Distance Compatible Set-Labeling of Graphs

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Abstract

Distance compatible set-labeling of a graph G is an injective set-assignment
$f : V(G) \to 2^X$, X a nonempty ground set, such that the corresponding induced function $f^{\oplus} : V(G) \times V(G) \to 2^X - \{\emptyset\}$, defined by $f^{\oplus}(u, v) = f(u) \oplus f(v)$ satisfies
$| f^{\oplus}(u, v) |= k^f_{(u,v)}\ d(u, v)$ for all distinct $u, v \in V(G)$, where $d(u, v)$ is the distance between u and v, and $k^f_{(u,v)}$ is a constant, not necessarily an integer. A dcsl f of a (p, q)-graph G is *dispersive* if the constants of proportionality k^f_{uv} with respect to f, $u \neq v$, $u, v \in V(G)$ are all distinct G is (k, r)-*arithmetic*, if the constants of proportionality with respect to f can be arranged in the arithmetic progression, $k, k + r, k + 2r, \ldots, k + (q - 1)r$ and if G admits such a dcsl then G is a (k, r)-*arithmetic dcsl-graph*. This paper present our investigations on these new notions.

Keywords: dispersive distance compatible set-labeling, edge-dispersive distance compatible set-labeling, (k, r)-arithmetic distance compatible set-labeling

1 Introduction

For all terminology and notation which are not defined in this paper, we refer the reader to F. Harary. Unless mentioned otherwise, all the graphs considered in this paper are finite, simple and without self-loops.
Acharya [1] introduced the notion of *set-valuation* as set analogue of number valuation. For a (p, q) graph $G = (V, E)$ and a nonempty set X of cardinality n, *Acharya* [**?**] defined *set-indexer* of G as an injective *set-valued* function $f : V(G) \to 2^X$ such that the function $f^{\oplus} : E(G) \to 2^X - \emptyset$ defined by

$f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where 2^X is the set of all subsets of X and "$\oplus$" is the *symmetric difference* of sets.
B. D. Acharya et.al [2] introduced the notion of distance-compatible set-labeling (dcsl) of a graph G as an injective set-assignment $f : V \to 2^X$, X being a nonempty 'ground set', such that the corresponding induced function $f^{\oplus} : V(G) \times V(G) \to 2^X - \emptyset$, defined by $f^{\oplus}(uv) = (f(u) - f(v)) \cup (f(v) - f(u))$ satisfies $| f^{\oplus}(u,v) |= k^f_{(u,v)} d(u,v)$ for all distinct $u, v \in V(G)$, where $d(u,v)$ is the distance between u and v and $k_{(u,v)}$ is a constant, not necessarily an integer. The main aim of this paper is to present an account of what we know of such graphs and their corresponding distance compatible set-labeling.

Definition 1.1. *[2] Let $G = (V,E)$ be any connected (p,q) graph. A distance compatible set-labeling of a graph G is an injective set-assignment $f : V(G) \to 2^X$, X a nonempty ground set, such that the corresponding induced function $f^{\oplus} : V(G) \times V(G) \to 2^X - \emptyset$, defined by $f^{\oplus}(uv) = f(u) \oplus f(v)$ satisfies $| f^{\oplus}(uv) |= k^f_{(u,v)} d(u,v)$ for all distinct $u, v \in V(G)$, where $d(u,v)$ is the distance between u and v and $k^f_{(u,v)}$ is a constant, not necessarily an integer. G is distance compatible set-labeled (dcsl) graph if it admits a dcsl. We denote a dcsl-graph G with a dcsl, f by the ordered pair (G, f). The corresponding ground set is called a dcsl-set.*

Definition 1.2. *A distance compatible set-labeling f of a (p,q)-graph G is* dispersive *if the constants of proportionality k^f_{uv} with respect to f, $u \neq v$, $u, v \in V(G)$ are all distinct and G is* dispersible *if it admits a dispersive dcsl.*

Definition 1.3. *A dispersive distance compatible set-labeling f of a (p,q)-graph G is (k,r)-*arithmetic*, if the constants of proportionality with respect to f can be arranged in the arithmetic progression, $k, k+r, k+2r, \ldots, k+(q-1)r$ and if G admits such a dcsl then G is a (k,r)-*arithmetic dcsl-graph.

Remark 1.4. *Given a dcsl-graph (G, f), its definition implies that $|f^{\oplus}(uv)| = k^f_{uv} d(u,v)$, for all $u, v \in V(G)$, $u \neq v$, where k^f_{uv}'s are the* constants of proportionality. *Let $\mathcal{K}_f(G) = \{k^f_{uv} : u \neq v, \ \ u, v \in V(G)\}$. Note that in the case of a dispersible dcsl-graph (G, f) of order n, $\mathcal{K}_f(G)$ contains $\frac{n(n+1)}{2}$ distinct numbers.*

2 Dispersible dcsl-graphs

As defined already, a dcsl f of a graph G is *dispersive* if the constants of proportionality k^f_{uv}, $u \neq v$, $u, v \in V(G)$ are all distinct and G is *dispersible* if it admits a dispersive dcsl. A natural question that arises is what are the classes of graphs which admits a dispersive dcsl. Theorem 2.1 is an attempt to answer this question.

Theorem 2.1. *K_n is dispersible for all $n \geq 1$.*

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Let $X = \{1, 2, \ldots, 2^{n-1}\}$. Define $f : V(K_n) \to 2^X$ by $f(v_i) = \{1, 2, 3, \ldots, 2^{i-1}\}$, $1 \leq i \leq n$. Clearly, $f(v_i) \subset f(v_j)$, for $i < j$. Now, we shall prove that the constants of proportionality k^f_{uv} are all distinct, for distinct $u, v \in V(K_n)$.

$$k^f_{(v_1,v_i)} = |f^{\oplus}(v_1 v_i)|, \quad 2 \leq i \leq n = \{1, 2^2 - 1, 2^3 - 1, \ldots, 2^{n-1} - 1\}$$

$$k^f_{(v_2,v_i)} = |f^{\oplus}(v_2 v_i)|, \quad 3 \leq i \leq n = \{2^2 - 2, 2^3 - 2, \ldots, 2^{n-1} - 2\}$$

$$k^f_{(v_3,v_i)} = |f^{\oplus}(v_3 v_i)|, \quad 4 \leq i \leq n = \{2^3 - 2^2, 2^4 - 2^2, \ldots, 2^{n-1} - 2^2\}$$

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$$k^f_{(v_{n-1},v_n)} = |f^{\oplus}(v_{n-1} v_n)| = 2^{n-1} - 2^{n-2}.$$

Hence, $k^f_{(u,v)}$ are distinct for all distinct $u, v \in V(K_n)$, whence the dcsl f of K_n is dispersive, so that K_n is dispersible dcsl-graph. □

We can prove that $K_{1,n}$ is dispersible for all finite values of n. Hence we have the following Theorem.

Theorem 2.2. *The star $K_{1,n}$ is dispersible dcsl-graph for all finite values of n.*

Proof. Let $V(K_1, n) = \{v_0, v_1, \ldots, v_n\}$. Let $X = \{1, 2, \ldots, 2^{n-1}\}$. Define $f : V(K_1, n) \to 2^X$ by $f(v_0) = \emptyset$
$f(v_i) = \{1, 2, 3, \ldots, 2^{i-1}\}$, $1 \leq i \leq n$. Now
$k^f_{(v_1,v_i)} = \{1, 2, 4, 8, ...2^{n-1}\}$ and
$k^f_{(v_i,v_j)} = \frac{2^{j-1} - 2^{i-1}}{2}$, $1 \leq i < j \leq n$. Also $k^f_{(v_i,v_j)} \neq k^f_{(v_k,v_l)}$, $1 \leq i < j \leq n$, $1 \leq k < l \leq n$. Therefore f is a dispersive dcsl. □

2.1 Edge-dispersible dcsl-graphs

As we have seen in the previous section all graphs need not be dispersible. However, to identify the characteristics of a graph model, sometimes it is enough to consider the constants of proportionality $k^f_{(u,v)}$ whenever uv is an edge of G. Hence, we define a new concept namely, edge-dispersible dcsl-graphs as follows.

Definition 2.3. *A dcsl f of a (p, q)-graph G is* edge-dispersive *if the constants of proportionality $k^f_{(u,v)}$: $uv \in E(G)$ are all distinct and G is edge-dispersible graph if it admits an edge- dispersive dcsl.*

Remark 2.4. *All dispersible dcsl-graphs are edge-dispersible. However the converse need not be true. We may observe that dispersible dcsl-graphs are subclass of edge-dispersible dcsl-graphs. In the case of complete graphs dispersibility and edge dispersibility are the same, since every pair of vertices in a complete graph is adjacent. Following results depicts some classes of edge-dispersible dcsl-graphs.*

Theorem 2.5. *The paths P_n are edge-dispersible dcsl-graphs.*

Proof. Let P_n be a path on n vertices. Let $v_0, v_1, \ldots, v_{n-1}$ be the n vertices of P_n. Let $X = \{1, 2, 3, \ldots, l\}$, $l = \frac{n(n-1)}{2}$.
Define $f : V(P_n) \to 2^X$ defined by
$f(v_0) = \emptyset$;
$f(v_1) = \{1\}$; and,
$f(v_i) = f(v_{i-1}) \cup \{max\ (f(v_{i-1})) + j,\ 1 \leq j \leq i\},\ 2 \leq i \leq n$. Then, $\mid f^{\oplus}(e_i) \mid = \mid f(v_{i-1}) \oplus f(v_i) \mid = \mid \{max f(v_{i-1}) + j,\ 1 \leq j \leq i\} \mid = i = i(d(v_{i-1}v_i))$. That is, $k^f_{(v_{i-1}, v_i)} = i$. Therefore, P_n is edge-dispersible. □

Since dispersible dcsl-graphs are subclass of edge-dispersible dcsl-graphs, all finite stars are edge-dispersible graphs by Theorem 2.2. Here we give a different labeling in which the cardinality of the dcsl set is less than that of the labeling given in Theorem 2.2.

Theorem 2.6. *The star $K_{1,n}$ of order $n+1$ is edge-dispersible dcsl-graphs.*

Proof. Consider $K_{1,n}$ with $n+1$ vertices. Let $V(K_{1,n}) = \{v_0, v_1, \ldots, v_n\}$.
Let $X = \{1, 2, 3, \ldots, k\}$, where $k = \frac{n(n+1)}{2}$.
Define $f : V(K_{1,n}) \to 2^X$ defined by
$f(v_0) = \emptyset$
$f(v_1) = \{1\}$ and,
$f(v_i) = \{max\ (f(v_{i-1})) + j,\ 1 \leq j \leq i\},\ 2 \leq i \leq n$. Then,
$\mid f^{e_i} \mid = \mid f(v_0) \oplus f(v_i) \mid = \mid \{max\ (f(v_{i-1})) + j,\ 1 \leq j \leq i\} \mid = i,\ 1 \leq i \leq n$.
That is, $k^f_{(v_0, v_i)} = i,\ 1i \leq n$. Therefore $K_{1,n}$ is edge-dispersible. □

Theorem 2.7. *Every complete bipartite graph is edge-dispersible.*

Proof. Let X and Y be the bipartition of the vertex set of $K_{m,n}$, where $V(X) = \{u_1, u_2, \ldots, u_m\}$
$V(Y) = \{v_1, v_2, \ldots, v_n\}$.
Let $X^* = \{1, 2, \ldots, mn\}$
Define $f : V(K_{m,n}) \to 2^{X^*}$, defined by
$f(u_1) = \emptyset$;
$f(u_i) = \{n+1, n+2, \ldots, in\}$, $2 \leq i \leq m$;
$f(v_j) = \{1, 2, \ldots, j\}$, $1 \leq j \leq n$. Then,
$\mid f^{\oplus}(u_1 v_j) \mid = \mid \{1, 2, \ldots, j\} \mid = j$ and, $\mid f^{\oplus}(u_i v_j) \mid = (i-1)n + j,\ 2 \leq i \leq m,\ 1 \leq j \leq n$. Now, $\mid f^{\oplus}(u_1 v_j) \mid = \mid f^{\oplus}(u_i v_j) \mid$ implies $j = (i-1)n + j$,

which implies $i = 1$, not possible since, $2 \leq i \leq m$. Thus $K_{m,n}$ is edge-dispersible. □

Remark 2.8. *By Theorem 2.1 complete graph K_n is edge-dispersible. It is interesting to note that in the case of complete graphs dispersibility and edge-dispersibility are one and the same.*

3 (k, r)-arithmetic dcsl-graphs

Again, recall that a dcsl f of a (p, q)-graph $G = (V, E)$ is (k, r)-*arithmetic* if the constants of proportionality with respect to f can be arranged in the arithmetic progression, $k, k+r, k+2r, \ldots, k+(q-1)r$ and if G admits such a dcsl then, G is a (k, r)-*arithmetic dcsl-graph*.

Theorem 3.1. *The path P_n is $(1, 1)$-arithmetic dcsl if $n \leq 3$.*

Proof. The subsets $\emptyset$, $\{1\}$ and $\{1, 2, 3\}$ of a set $X = \{1, 2, 3\}$, respectively assigned to the vertices v_1, v_2 and v_3 of P_3, we get a $(1, 1)$-arithmetic dcsl-graph. Also $\emptyset$ *and* $\{1\}$ gives the $(1, 1)$-arithmetic labeling of P_2 and $\emptyset$ gives a $(1, 1)$-arithmetic labeling of P_1. □

Remark 3.2. *We proved that paths P_n are $(1, 1)$-arithmetic dcsl-graphs if $n \leq 3$. We strongly believe that paths are not $(1, 1)$-arithmetic dcsl-graphs for higher values of n. Thus we pose the Conjecture 1.*

Conjecture 1. *Path P_n is $(1, 1)$-arithmetic dcsl-graph if and only if $n \leq 3$.*

In this section, we also establish that the complete graph K_n admits a (k, r)-arithmetic dcsl if and only if $n \leq 4$ and $k = r$.

Theorem 3.3. *The complete graph K_n is a (k, r)-arithmetic dcsl-graph if and only if $n \leq 4$ and $k = r$.*

Proof. **Necessity:** Let K_n admit a (k, r)-arithmetic dcsl $f : V(K_n) \rightarrow 2^X$, where $X = \{x_1, x_2, \ldots, \}$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. We shall prove the theorem in two Parts.
PART 1: In this part, we prove that if K_n is (k, r)-arithmetic dcsl, then $k = r$.
We need to prove this in two cases, namely when the empty set $\emptyset$ is or is not assigned to a vertex of K_n.
Case1: $f(v_i) = \emptyset$ for some $v_i \in V(K_n)$.
Without loss of generality, assume $f(v_1) = \emptyset$. Since f is a (k, r)-arithmetic dcsl, there exists an edge say $v_1 v_j \in E(K_n)$ such that $|f^{\oplus}(v_1 v_j)| = k$. Without loss of generality, assume that $|f^{\oplus}(v_1 v_2)| = k$ so that $f(v_2) = \{x_1, x_2, \ldots, x_k\}$. Again, since f is a (k, r)-arithmetic dcsl there exists an edge in K_n such that

the cardinality of the symmetric difference of the subsets of X assigned to its end vertices is $k+r$. For this, there are two possibilities viz., $|f^{\oplus}(v_1v_t)| = k+r$ or $|f^{\oplus}(v_2v_t)| = k+r$. Without loss of generality, assume $v_t = v_3$. Then, we have either $|f^{\oplus}(v_2v_3)| = k+r$ or $|f^{\oplus}(v_1v_3)| = k+r$.

First, we consider the possibility that $|f^{\oplus}(v_1v_3)| = k+r$**.**

Then, we have the following possibilities.

$f(v_3) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_r\}$

or $f(v_3) = \{y_1, y_2, \ldots, y_{k+r}\}$.

When $f(v_3) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_r\}$, we get $|f^{\oplus}(v_2v_3)| = r$, which is not possible. Hence, $f(v_3) = \{y_1, y_2, \ldots, y_{k+r}\}$. This implies $|f^{\oplus}(v_2v_3)| = k+k+r = 2k+r$.

Therefore, the only possibility of getting $|f^{\oplus}(v_2v_3)| = k+2r$ occurs when $k+2r = 2k+r$, which implies $k = r$.

Next, we consider the possibility that $|f^{\oplus}(v_2v_3)| = k+r$**.**

In this case, there arise the following possibilities.

$f(v_3) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{k+r}\}$

or $f(v_3) = \{y_1, y_2, \ldots, y_r\}$

When $f(v_3) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{k+r}\}$, $|f^{\oplus}(v_1v_3)| = 2k+r$; and when $f(v_3) = \{y_1, y_2, \ldots, y_r\}$, $|f^{\oplus}(v_1v_3)| = r$, which is not possible. Hence, $f(v_3) = \{y_1, y_2, \ldots, y_{k+r}\}$. Therefore, the only possibility of getting $|f^{\oplus}(v_1v_3)| = k+2r$ occurs when $k+2r = 2k+r$, which implies $k = r$.

Case 2: $f(v_i) \neq \emptyset$ for any $v_i \in V(K_n)$.

Without loss of generality, assume $f(v_1) = \{x_1, x_2, \ldots, x_t\}$. Since f is a (k, r)-arithmetic dcsl there exists an edge say $v_1v_j \in E(K_n)$ such that $|f^{\oplus}(v_1v_j)| = k$. Without loss of generality, assume that $|f^{\oplus}(v_1v_2)| = k$ so that $f(v_2) = \{x_1, x_2, \ldots, x_{t+k}\}$ or $f(v_2) = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_k\}$. Again, since f is a (k, r)-arithmetic dcsl, there exists an edge in K_n such that the cardinality of the symmetric difference of the subsets of X assigned to its end vertices is $k+r$. For this, there are two possibilities viz., $|f^{\oplus}(v_1v_t)| = k+r$ or $|f^{\oplus}(v_2v_t)| = k+r$. Without loss of generality, assume $v_t = v_3$. Thus, we have either $|f^{\oplus}(v_2v_3)| = k+r$ or $|f^{\oplus}(v_1v_3)| = k+r$.

Suppose $f(v_2) = \{x_1, x_2, \ldots, x_{t+k}\}$. If $|f^{\oplus}(v_2v_3)| = k+r$, then we have following possible assignments for the vertex v_3.

(i) $f(v_3) = \{x_1, x_2, \ldots, x_{t+k}, y_1, y_2, \ldots, y_{k+r}\}$ or

(ii) $f(v_3) = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_r\}$ or

(iii) $f(v_3) = \{y_1, y_2, \ldots, y_{r-t}\}$

or (iv) $f(v_3) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{k+r-t}\}$

(i) implies $|f^{\oplus}(v_1v_3)| = 2k+r$;

(ii) implies $|f^{\oplus}(v_1v_3)| = t+r$, which is not possible.

(iii) implies $|f^{\oplus}(v_1v_3)| = t+r-t = r$, which is not possible.

(iv) implies $|f^{\oplus}(v_1v_3)| = 2k+r$.

Therefore, the only possibility of getting $|f^{\oplus}(v_1v_3)| = k+2r$ occurs when $k+2r = 2k+r$, which implies $k = r$.

The proof of the case when $|f^{\oplus}(v_1v_3)| = k+r$ follows in a similar way.
Next, suppose $f(v_2) = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_k\}$. We should then have either $|f^{\oplus}(v_2v_3)| = k+r$ or $|f^{\oplus}(v_1v_3)| = k+r$. Assume $|f^{\oplus}(v_2v_3)| = k+r$ whence we have the following possibilities.
(i) $f(v_3) = \{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_k, z_1, z_2, \ldots z_{k+r}\}$ or
(ii) $f(v_3) = \{x_1, x_2, \ldots, x_t, z_1, z_2, \ldots, z_r\}$ or
(iii) $f(v_3) = \{y_1, y_2, \ldots, y_k, z_1, z_2, \ldots, z_{k+r-t}\}$ or
(iv) $f(v_3) = \{z_1, z_2, \ldots, z_{r-t}\}$
(i) implies $|f^{\oplus}(v_1v_3)| = 2k+r$;
(ii) implies $|f^{\oplus}(v_1v_3)| = r$, which is not possible.
(iii) implies $|f^{\oplus}(v_1v_3)| = 2k+r$, which is not possible.
(iv) implies $|f^{\oplus}(v_1v_3)| = r$.
Therefore, the only possibility of getting $|f^{\oplus}(v_1v_3)| = k+2r$ occurs when $k+2r = 2k+r$, which implies $k = r$.
Hence, if $K_n, n \geq 3$ is (k,r)-arithmetic dcsl, then $k = r$.
PART 2: In this part, we prove that if K_n is (r,r)-arithmetic dcsl, then $n \leq 4$. If possible, suppose $n > 4$ and K_n is (r,r)-arithmetic dcsl with an (r,r)-arithmetic dcsl f. Hence, the constants of proportionality with respect to f can be arranged in the arithmetic progression, $r, 2r, 3r, 4r, 5r, 6r, 7r, 8r, 9r, 10r, \ldots$. Without loss of generality, assume $f(v_1) = X_1 = \{x_1, x_2, \ldots, x_k\}$. Then, to have the constant of proportionality r on one of the edges of K_5 we have the following selection of the set X_2, say at the vertex v_2;
(1) $f(v_2) = X_2 = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{2k+r}\}$ or
(2) $f(v_2) = X_2 = \{x_1, x_2, \ldots, x_k\} \cup \{y_1, y_2, \ldots, y_{k+r}\}$.
Now, in order to get the constant of proportionally $2r$ on one of the other edges of K_n, we have the following assignment of subsets of the ground set X to the vertices say at v_3.
(1.a1) $f(v_3) = X_3 = \{x_1, x_2, \ldots, x_k, x_{k+1} \ldots, x_{k+2r}\}$;
(1.a2) $f(v_3) = X_3 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}\}$;
(1.a3) $f(v_3) = X_3 = \{x_1, x_2, \ldots, x_{k+r}, y_1, y_2, \ldots, y_r\}$.
Consider (1.a1): Then,
$|f(v_1) \oplus f(v_2)| = r$;
$|f(v_1) \oplus f(v_3)| = 2r$ and $|f(v_2) \oplus f(v_3)| = r$, a contradiction to the assumption that f is (r,r)-dcsl. Hence, the labeling (1.a1) is not admissible.
Consider (1.a2) Then, $|f(v_1) \oplus f(v_2)| = r$;
$|f(v_1) \oplus f(v_3)| = 2r$ and $|f(v_2) \oplus f(v_3)| = 3r$, which are admissible.
Consider (1.a3) Then,
$|f(v_1) \oplus f(v_2)| = r$;
$|f(v_1) \oplus f(v_3)| = 2r$ and $|f(v_2) \oplus f(v_3)| = r$, a contradiction to the assumption that f is (r,r)-dcsl. Therefore, the labeling (1.a3) is not admissible.
Hence, until at this stage, we have the admissible assignment of subsets of X as follows:
$f(v_1) = X_1 = \{x_1, x_2, \ldots, x_k\}$

$f(v_2) = X_2 = \{x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_{2k+r}\}$
$f(v_3) = X_3 = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{2r}\}$
By symmetry, the possibility of the assignment (2), $f(v_2) = X_2 = \{x_1, x_2, \dots, x_k\} \cup \{y_1, y_2, \dots, y_{k+r}\}$, reduces to the same choice namely,
$f(v_3) = X_3 = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{2r}\}$.
Hence, without loss of generality, assume
$f(v_3) = X_3 = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{2r}\}$.
Now, to have the constant of proportionality $4r$ on pairs of vertices involving v_1, v_2, v_3 we have the following three possible selections.
II.a. $f(v_4) = X_4 = \{x_1, x_2, \dots, x_{k+4r}\}$ or
II.b. $f(v_4) = X_4 = \{x_1, x_2, \dots, x_k, x_{k+r}, \dots, y_1, y_2, \dots, y_{3r}\}$ or
II.c. $f(v_4) = X_4 = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{2r}, t_1, t_2, \dots, t_{2r}\}$.
Consider (II.a): In this case, we get
$|f(v_1) \oplus f(v_2)| = r$;
$|f(v_1) \oplus f(v_3)| = 2r$; $|f(v_2) \oplus f(v_3)| = 3r$; $|f(v_1) \oplus f(v_4)| = 4r$; $|f(v_2) \oplus f(v_4)| = 3r$, a contradiction to our assumption that f is (r, r)-dcsl. Hence, the labeling (1.a1) is not admissible.
Consider (II.b): In this case,
$|f(v_1) \oplus f(v_2)| = r$;
$|f(v_1) \oplus f(v_3)| = 2r$; $|f(v_2) \oplus f(v_3)| = 3r$; $|f(v_1) \oplus f(v_4)| = 4r$; $|f(v_2) \oplus f(v_4)| = 3r$, a contradiction to the assumption that f is (r, r)-dcsl. Hence, the labeling (1.a1) is not admissible.
Consider (II.c): In this case, $|f(v_1) \oplus f(v_2)| = r$;
$|f(v_1) \oplus f(v_3)| = 2r$; $|f(v_2) \oplus f(v_3)| = 3r$; $|f(v_1) \oplus f(v_4)| = 4r$; $|f(v_2) \oplus f(v_4)| = 5r$ and $|f(v_3) \oplus f(v_4)| = 6r$, whence (1.a1) is admissible.
Hence, until at this stage of completing the assignment of the vertices v_1, v_2, v_3, v_4, we have the admissible assignments as follows:
$f(v_1) = X_1 = \{x_1, x_2, \dots, x_k\}$
$f(v_2) = X_2 = \{x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_{2k+r}\}$
$f(v_3) = X_3 = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{2r}\}$.
$f(v_4) = X_4 = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{2r}, t_1, t_2, \dots, t_{2r}\}$.
Now, to have the constant of proportionality $7r$ on pairs with the vertices v_1, v_2, v_3, v_4 we have the following four selections.
II.a. With the vertex v_1, $f(v_5) = X_5 = \{x_1, x_2, \dots, x_{k+7r}\}$ or
II.b. With the vertex v_2:
II.b1. $f(v_5) = X_5 = \{x_1, x_2, \dots, x_k, x_{k+8r}, \dots, y_1, y_2, \dots, y_{3r}\}$
or
II.b2. $f(v_5) = X_5 = \{x_1, x_2, \dots, x_k, x_{k+r}, \dots, y_1, y_2, \dots, y_{6r}\}$
II.c. With v_3: II.c1. $f(v_5) = X_5 = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{9r}\}$.
or
II.c2. $f(v_5) = X_5 = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{2r}, h_1, h_2, \dots, h_{7r}\}$
II.d. With v_4:
II.d1. $f(v_5) = X_5 = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{2r}, t_1, t_2, \dots, t_{9r}\}$.

or
II.d2. $f(v_5) = X_5 = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{2r}, t_1, t_2, \ldots, t_{2r}, z_1, z_2, \ldots, z_{7r}\}$
Consider II.a. Then, we get
$|f(v_1) \oplus f(v_2)| = r$;
$|f(v_1) \oplus f(v_3)| = 2r$; $|f(v_2) \oplus f(v_3)| = 3r$; $|f(v_1) \oplus f(v_4)| = 4r$; $|f(v_2) \oplus f(v_4)| = 5r$; $|f(v_3) \oplus f(v_4)| = 6r$; $|f(v_2) \oplus f(v_4)| = 6r = |f(v_3) \oplus f(v_4)|$, a contradiction to the assumption that f is (r, r)-dcsl. Hence, the labeling (II.a) is not admissible.
Consider II.b1. Then, we get
$|f(v_1) \oplus f(v_2)| = r$;
$|f(v_1) \oplus f(v_3)| = 2r$; $|f(v_2) \oplus f(v_3)| = 3r$; $|f(v_1) \oplus f(v_4)| = 4r$; $|f(v_2) \oplus f(v_4)| = 5r$; $|f(v_3) \oplus f(v_4)| = 6r$; $|f(v_1) \oplus f(v_5)| = 8r$; $|f(v_2) \oplus f(v_5)| = 7r$; $|f(v_3) \oplus f(v_5)| = 10r$; $|f(v_4) \oplus f(v_5)| = 12r$, a contradiction to the assumption that f is (r, r)-dcsl. Hence, the labeling (II.b1) is not admissible.
Consider II.b2. Then, we get
$|f(v_1) \oplus f(v_2)| = r$;
$|f(v_1) \oplus f(v_3)| = 2r$; $|f(v_2) \oplus f(v_3)| = 3r$; $|f(v_1) \oplus f(v_4)| = 4r$; $|f(v_2) \oplus f(v_4)| = 5r$; $|f(v_3) \oplus f(v_4)| = 6r$; $|f(v_2) \oplus f(v_5)| = 6r = |f(v_3) \oplus f(v_4)|$, a contradiction to the assumption that f is (r, r)-dcsl. Hence, the labeling (II.b2) is not admissible.
Consider II.c1. Then, we get
$|f(v_1) \oplus f(v_2)| = r$;
$|f(v_1) \oplus f(v_3)| = 2r$; $|f(v_2) \oplus f(v_3)| = 3r$; $|f(v_1) \oplus f(v_4)| = 4r$; $|f(v_2) \oplus f(v_4)| = 5r$; $|f(v_3) \oplus f(v_4)| = 6r$; $|f(v_1) \oplus f(v_5)| = 11r$, which is greater than $10r$, the maximum constant of proportionality and hence a contradiction to the assumption that f is (r, r)-dcsl. Hence, the labeling (II.c1) is not admissible.
Consider II.c2. Then, we get
$|f(v_1) \oplus f(v_2)| = r$;
$|f(v_1) \oplus f(v_3)| = 2r$; $|f(v_2) \oplus f(v_3)| = 3r$; $|f(v_1) \oplus f(v_4)| = 4r$; $|f(v_2) \oplus f(v_4)| = 5r$; $|f(v_3) \oplus f(v_4)| = 6r$; $|f(v_1) \oplus f(v_5)| = 9r$; $|f(v_2) \oplus f(v_5)| = 10r$; $|f(v_3) \oplus f(v_5)| = 7r$; $|f(v_4) \oplus f(v_5)| = 9r = |f(v_1) \oplus f(v_5)|$, a contradiction to the assumption that f is (r, r)-dcsl. Hence, the labeling (II.c2) is not admissible.
Hence, all the possible choices of assignment of the subsets of X to the vertices fail to define f as an (r, r)-dcsl. Hence, for $n \geq 5$, there exists no (r, r)-dcsl for K_n.
Sufficiency: Let $n \leq 4$ and $k = r$. We then display an (r, r)-arithmetic dcsl for K_n for each n.
For $n = 1, 2$, assign $f(v_1) = \emptyset$, and $f(v_1) = \emptyset$, $f(v_2) = \{x_1, x_2, \ldots, x_r\}$ respectively. In each case, it is easy to see that f so defined is indeed an (r, r)-arithmetic dcsl of K_n.
Next, let $V(K_3) = \{v_1, v_2, v_3\}$ and let $f : V(K_3) \to 2^X$ be defined by $f(v_1) =$

$X_1 = \{x_1, x_2, \ldots, x_r\}$; $f(v_2) = X_2 = \{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_{2r}\}$, $x_i \neq y_i$; $f(v_3) = X_3 = \{x_1, x_2, \ldots, x_r, x_{r+1} \ldots, x_{2r}\}$. Then, since K_3 is complete, $d(v_i, v_j) = 1$ for all distinct $i, j \in \{1, 2, 3\}$. Also, $|f(v_1) \oplus f(v_2)| = r$; $|f(v_1) \oplus f(v_3)| = 2r$ and $|f(v_2) \oplus f(v_3)| = 3r$. Thus, K_3 is an (r, r)-arithmetic dcsl-graph.
Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$. Define $f : V(K_4) \rightarrow 2^X$ so that $f(v_1) = X_1 = \{x_1, x_2, \ldots, x_r\}$; $f(v_2) = X_2 = \{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_{2r}\}$, $x_i \neq y_i$; $f(v_3) = X_3 = \{x_1, x_2, \ldots, x_r, x_{r+1} \ldots, x_{2r}\}$ and
$f(v_4) = X_4 = \{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_{2r}, t_1, t_2, \ldots, t_{2r}\}$. Since K_4 is complete,
$d(v_i, v_j) = 1$ for all distinct $i, j \in \{1, 2, 3, 4\}$. Then,
$|f(v_1) \oplus f(v_2)| = r$; $|f(v_1) \oplus f(v_3)| = 2r$; $|f(v_1) \oplus f(v_4)| = 4r$;
$|f(v_2) \oplus f(v_3)| = 3r$; $|f(v_2) \oplus f(v_4)| = 5r$ and $|f(v_3) \oplus f(v_4)| = 6r$.
Hence, $\mathcal{K}_f(K_4) = \{r, 2r, 3r, 4r, 5r, 6r\}$ and the proof is complete. □

4 Scope and Conclusion

Apart from theoretical interest, distance compatible set-labeling (dcsl) has found applications in many practically interesting areas such as Quantitative Structure-Activity Relation (QSAR) in discrete mathematical chemistry (see S.C. Basak *et. al.*, [3]) and, studies on the effect of indirect qualitative relationships between individuals in a social network (see Fiksel [5] and Kovchegov [8]). Characterization of dispersible dcsl-graph and finding the *dispersivity* $\nu(G)$ of G (the least cardinality of a ground set X such that G admits a dispersive dcsl) are interesting problems for further investigations. It is interesting to check whether there exist a graphical realization of an arithmetic progression in the set of real numbers or integers.

Acknowledgements
The authors are thankful to the Department of Science & Technology, Government of India for supporting this research under the project No. SR/S4/MS:277/06.

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Received: March, 2010