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Decomposable Abelian Groupoids

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Abstract. We study harmonic analysis on decomposable abelian groupoids. We show that an abelian r-discrete groupoid with finite unit space and its dual groupoid are decomposable. For a decomposable abelian groupoid, we characterize the dual groupoid and show that the corresponding C^*-algebra is CCR and has a Hausdorff spectrum. We also show that Glimm ideals of these C^*-algebras are maximal.

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1. Introduction

In this paper we study C^*-algebras on decomposable abelian groupoids. The structure of abelian groupoids is recently studied by the first author in [18]. Our basic reference for groupoids is [19].

A groupoid G is a small category in which each morphism is invertible. The unit space $X = G^{(0)}$ of G is the subset of elements $\gamma\gamma^{-1}$ where γ ranges over G. The range map $r : G \to G^{(0)}$ and source map $s : G \to G^{(0)}$ are defined by $r(\gamma) = \gamma\gamma^{-1}$, $s(\gamma) = \gamma^{-1}\gamma$, for $\gamma \in G$. For $u \in G^{(0)}$, we set $G^u = r^{-1}(u)$ and $G_u = s^{-1}(u)$. The loop space $G^u_u = \{\gamma \in G | r(\gamma) = s(\gamma)\}$ is called the *isotropy group* of G (at u).

Definition 1.1. An *abelian groupoid* is a groupoid whose isotropy groups are abelian.

A continuous open map $\psi : G \to H$ between topological groupoids is called a *quasilocal* homeomorphism if it is locally injective, namely the collection of open sets $U(\psi) = \{U \subset G : \psi|U \text{ is injective}\}$ is a basis (of the topology of) G. ψ is called a *local* (*étale*) homeomorphism if every point $x \in G$ has an open neighborhood U such that the restriction of f to U is a homeomorphism from U onto $f(U)$. Note that a surjective quasilocal homeomorphism is a local homeomorphism.

A topological groupoid G is called r-*discrete* if it has a cover of open G-sets [19]. In this case, the counting measures on fibres G^x (which are countable) [19, Lemma I.2.7] form a continuous Haar system.

There is a notion of topological AF equivalence relation which is the analog of a hyperfinite measured equivalence relation [20]. For topological groupoids, AF equivalence relations defined in [20, Definition 3.1 (iii)] are called AF groupoids in [19, Definition 1.1 page 123]. There are other equivalent definitions. A convenient one is to say that it is given by a Bratteli diagram; more precisely, it can be realized as the tail equivalence relation on the space of infinite paths of a Bratteli diagram.

Definition 1.2 ([19, 20])**.** An (r-discrete) equivalence relation R on X is

(i) *proper* if its graph R is closed in $X \times X$ and the quotient map $q : X \to X/R$ is a local homeomorphism;

(ii) *approximately proper* (AP) if $R = \bigcup_n R_n$ where (R_n) is an increasing sequence of proper equivalence relations;

(iii) *approximately finite* (AF) if it is AP and X is totally disconnected.

AF groupoids are characterized in [19, page 123]. The inductive limit topology turns an AP equivalence relation into an r-discrete locally compact Hausdorff groupoid [19, page 122]. Unless we specify otherwise, we equip R with this topology. Note that in part (*iii*) above, we may assume that R_n is open in R_{n+1}, and their Haar systems are compatible [2, page 22].

Remark 1.3. If the equivalence relation R on X is proper and r-discrete (the quotient space is Hausdorff and its quotient map $X \to X/R$ is a local homeomorphism, see [21, Definition 2.1]), then the groupoid R is proper in the sense of [1, page 14] (that is, the inverse image of every compact subset of $X \times X$ is compact in R).

The equivalence relation R is called *finite* if it is compact. If R is finite, then X is necessarily compact and X/R is Hausdorff and the map $\psi : X \to X/R$ is a surjective local homeomorphism, because ψ is open [9, page 976] and $R = R(\psi)$, where we define $R(\psi) = \{(z_1, z_2)|\psi(z_1) = \psi(z_2)\}$. An equivalence relation R on a compact space X is called *hyperfinite* if there is an increasing sequence of finite relations $R_n \subset R$, $n \geq 1$, such that $R = \bigcup_n R_n$.

Note that an equivalence relation R on X is approximately proper if there exists a sequence

$$X_0 \xrightarrow{\pi_{1,0}} X_1 \to \cdots \to X_{n-1} \xrightarrow{\pi_{n,n-1}} X_n \to \dots$$

where $X_0 = X$ and for each n, X_n is a Hausdorff space and $\pi_{n,n-1}$ is a surjective local homeomorphism such that

$$R = \{(x, y) \in X \times X : \exists n \in \mathbb{N}\ \pi_n(x) = \pi_n(y)\}$$

where $\pi_n = \pi_{n,n-1} \circ \pi_{n-1,n-2} \cdots \circ \pi_{1,0}$. Here we let R_n be the equivalence relation on X defined by π_n.

Definition 1.4. An abelian groupoid Γ is called *decomposable* if $R(\Gamma)$ is proper and r-discrete and has a basis consisting of compact open R-sets (bisections).

Note that decomposable abelian groupoids are amenable by [7, prop. 1.110], because proper principal groupoid $R(\Gamma)$ is amenable [18, Remark 1.2.22] and isotropy groups are abelian.

Lemma 1.5. *For a decomposable abelian groupoid* Γ*, the isotropy subgroupoid* $\Gamma(X)$ *is open.*

Proof. Since X is open in Γ and $r \times s$ is a continuous map, $\Gamma(X) = (r \times s)^{-1}(X)$ is open in Γ, because X is open in $R(\Gamma)$. □

Corollary 1.6. *Every decomposable abelian groupoid with a continuous Haar system is r-discrete.*

The pair $(\mathcal{C}, \mathcal{D})$ is called a *regular C^*-inclusion* if $\mathcal{D}$ is a maximal abelian C*-subalgebra of the unital C^*-algebra $\mathcal{C}$ such that $1 \in \mathcal{D}$ and

(i) $\mathcal{D}$ has the extension property in $\mathcal{C}$;

(ii) $\mathcal{C}$ is regular (as a $\mathcal{D}$-bimodule).

Let P denote the (unique) conditional expectation of $\mathcal{C}$ onto $\mathcal{D}$. We call $(\mathcal{C}, \mathcal{D})$ a *C^*-diagonal* if in addition,

(iii) P is faithful.

If $\mathcal{C}$ is non-unital, $\mathcal{D}$ is called a diagonal in $\mathcal{C}$ if its unitization $\widetilde{\mathcal{D}}$ is diagonal in $\widetilde{\mathcal{C}}$. We shall refer to $(\mathcal{C}, \mathcal{D})$ as a *diagonal pair.*

A *twist* $\mathcal{G}$ is a proper $\mathbb{T}$-groupoid such that $\mathcal{G}/\mathbb{T}$ is an r-discrete equivalence relation $\mathcal{R}$ [9]. There is a one-to-one correspondence between twists and diagonal pairs of C*-algebras: Define

$$C_c(\mathcal{R}, \mathcal{G}) = \{f \in C_c(\mathcal{G}) : f(t\gamma) = tf(\gamma)\,(t \in \mathbb{T}, \gamma \in \mathcal{G})\}.$$

Then $\mathcal{G}$ is a $\mathbb{T}$-groupoid over $\mathcal{R}$ and

$$D(\mathcal{G}) = \{f \in C_c(\mathcal{R}, \mathcal{G}) : \mathit{supp} f \subset \mathbb{T}\mathcal{G}^0\}$$

where $\mathbb{T}\mathcal{G}^0$ denotes the isotropy group bundle of $\mathcal{G}$.

$C_c(\mathcal{R}, \mathcal{G})$ is a C^*-algebra with a distinguished abelian subalgebra $D(\mathcal{G})$. Furthermore, $D(\mathcal{G}) \cong C_0(\mathcal{G}^{(0)})$. Now $C_c(\mathcal{R}, \mathcal{G})$ becomes a pre-Hilbert $D(\mathcal{G})$-module, whose completion $\mathbf{H}(\mathcal{G})$ is a Hilbert $C_0(\mathcal{G}^{(0)})$-module. Kumjian has constructed a $*$-homomorphism $\pi : C_c(\mathcal{R}, \mathcal{G}) \to \mathcal{B}(\mathbf{H}(\mathcal{G}))$ such that $\pi(f)g = f * g$ (the convolution product) for all $f, g \in C_c(\mathcal{R}, \mathcal{G})$ and defined $\mathcal{C}(\mathcal{G})$ to be the closure of $\pi(C_c(\mathcal{R}, \mathcal{G}))$ and $\mathcal{D}(\mathcal{G})$ to be the closure of $\pi(D(\mathcal{G}))$. It turns out that $\mathcal{D}(\mathcal{G})$ is diagonal in $\mathcal{C}(\mathcal{G})$, thus every twist gives rise to a diagonal pair of

C^*-algebras. Note that in [9, page 976] the notation $E(\mathcal{G})$ is used instead of $C_c(\mathcal{R},\mathcal{G})$ used in [14, page 129].

Conversely, given C*-algebras $\mathcal{C}$ and $\mathcal{D}$ such that $\mathcal{D}$ is a diagonal in $\mathcal{C}$, Kumjian gave a twist $\mathcal{G}$ such that $\mathcal{C}=\mathcal{C}(\mathcal{G})$ and $\mathcal{D}=\mathcal{D}(\mathcal{G})$, giving a bijective correspondence between twists and diagonal pairs [9].

2. Decomposable abelian groupoids

This section is devoted to decomposable abelian groupoids. We investigate properties of the C^*-algebras of these groupoids.

Theorem 2.1. *If Γ is a decomposable abelian groupoid, then $C^*(\Gamma)$ has continuous trace.*

Proof. By [16, Theorem 1.1], since the isotropy subgroupoid of Γ has a continuous Haar system (the map $x \mapsto \Gamma(x)$ is continuous on X) and the action of R on X is proper [16], the groupoid C^*-algebra $C^*(\Gamma)$ has continuous trace. □

A triple (A,Γ,σ) is called a *C^*-groupoid dynamical system* if A is a C^*-algebra, Γ is a locally compact groupoid with a faithful transverse function λ and $\sigma:\Gamma\to Aut(A)$ is a continuous homomorphism.

Theorem 2.2. *The triplet $(C^*(\Gamma(X)),R(\Gamma),\sigma)$ is a C^*-groupoid dynamical system, where Γ is a decomposable abelian groupoid and $\sigma: R\to Aut(C^*(\Gamma(X)))$ is a continuous homomorphism.*

Proof. As in [6], σ is a homomorphism from $R(\Gamma)$ into the isomorphism groupoid $Iso(C^*(\Gamma(X)))$ of $C^*(\Gamma(X))$, which is a separable C^*-algebras over the unit space X of $R(\Gamma)$. We write p for the projection of $C^*(\Gamma(X))$ onto X. In particular, $\sigma_k: C^*(\Gamma(s(k)))\to C^*(\Gamma(r(k)))$ is a C^*-isomorphism, for each $k\in R(\Gamma)$, $\sigma_k\varphi(\gamma)=\varphi(k\gamma k^{-1})$.

Let $\mathrm{s}^*(C^*(\Gamma(X)))$ and $\mathrm{r}^*(C^*(\Gamma(X)))$ be the bundles on $R(\Gamma)$ obtained by pulling back $C^*(\Gamma(X))$ via s and r, so that $\mathrm{s}^*(C^*(\Gamma(X)))=\{(k,\varphi):\varphi\in C^*(\Gamma(s(k)))\}$ and similarly for $\mathrm{r}^*(C^*(\Gamma(X)))$. Then σ determines a bundle map $\sigma^*: s^*(C^*(\Gamma(X)))\to r^*(C^*(\Gamma(X)))$ by the formula $\sigma^*(k,\varphi)=(k,\sigma_k(\varphi))$. Our continuity assumption means that for each continuous section f of $\mathrm{s}^*(C^*(\Gamma(X)))$, $\sigma^*\circ f$ is a continuous section of $\mathrm{r}^*(C^*(\Gamma(X)))$.

Let $C_c(R;r^*(C^*(\Gamma(X))))$ denote the space of continuous sections of $\mathrm{r}^*(C^*(\Gamma(X)))$ with compact support and, for $f,g\in C_c(R;r^*(C^*(\Gamma(X))))$, set

$$f*g(k)=\int f(t)\sigma_t(g(t^{-1}k))d\alpha^{r(k)}(t)$$

and $f^*(k)=\sigma_k(f(k^{-1})^*)$ [12, Definition 2.2]. Then $C_c(R,r^*(C^*(\Gamma(X))))$ is a topological $*$-algebra in the inductive limit topology to which Renault's disintegration theorem applies. □

The enveloping C*-algebra of $C_c(R,r^*(C^*(\Gamma(X))))$ is called the crossed product of R acting on $C^*(\Gamma(X))$ and is denoted $C^*(R,C^*(\Gamma(X)))$. It is easy to see that $C^*(X, C^*(\Gamma(X))) = C^*(\Gamma(X))$ [18, Cor. 1.3.8] and therefore it is a subalgebra of $C^*(R, C^*(\Gamma(X)))$. We shall denote the spectrum of $C^*(\Gamma(X))$ by $\mathcal{Z}$. Observe that $\mathcal{Z}$ may be expressed as the disjoint union of spectra $\bigsqcup_{x\in X}\widehat{\Gamma(x)}$, since $C^*(\Gamma(X)) = C_0(\mathcal{Z})$, and the natural projection $\hat{p}$ from $\mathcal{Z}$ to X is continuous and open [6]. The groupoid R acts on $\mathcal{Z}$ as follows. If $z \in \mathcal{Z}$, we shall write $z = [\pi_z]$ in order to specify a particular irreducible representation in the equivalence class represented by z. If

$$\mathcal{Z} \rtimes_c R = \{(x, k) \in \mathcal{Z} \times R : \hat{p}(z) = r(k)\}$$

and $(x, k) \in \mathcal{Z} \rtimes_c R$, then $z.k$ is defined to be $[\pi_z \circ \sigma_k]$. The action ρ of R on $\mathcal{Z}$ is well defined and continuous.

For $z \in \mathcal{Z}$, we shall write $\mathfrak{K}(z)$ for the quotient $C^*(\Gamma(\hat{p}(z)))/ker(\pi_z)$. Clearly $\mathfrak{K}(z)$ is well defined (it is independent of the choice of π_z) and is an elementary C^*-algebra. The collection $\{\mathfrak{K}(z)\}_{z\in\mathcal{Z}}$ may be given the structure of an elementary C*-algebra bundle over $\mathcal{Z}$ satisfying Fell's condition [11]. As a result, $C^*(\Gamma(X))$ is naturally isomorphic to $C_0(\mathcal{Z}, \mathfrak{K})$. The action of R on $\mathcal{Z}$ induces one on $\mathfrak{K}$ that we shall use here. We find it preferable to express this in terms of the action groupoid $\mathcal{Z} \rtimes_c R$ defined as follows:

$$(z, k)(zk, \eta) = (z, k\eta),\ (z, k)^{-1} = (zk, k^{-1}).$$

Note in particular that the unit space of $\mathcal{Z} \rtimes_c R$ may be identified with $\mathcal{Z}$ via $(z, \hat{p}(z)) \leftrightarrow z$. The range and source maps on $\mathcal{Z} \rtimes_c R$, denoted by $\widetilde{r}$ and $\widetilde{s}$, are given by $\widetilde{r}(z, k) = (z, r(k))$ and $\widetilde{s}(z, k) = (zk, s(k))$.

The groupoid $\mathcal{Z} \rtimes_c R$ in the product topology is locally compact, Hausdorff, and separable. It has a Haar system $\{\tilde{\lambda}^z\}_{z\in\mathcal{Z}}$ given by $\widetilde{\alpha}^z = \delta_z \times \alpha^{\hat{p}(z)}$. Observe that the action of R on $\mathcal{Z}$ is free (resp. proper) iff $\mathcal{Z} \rtimes_c R$ is principal (resp. proper).

The groupoid $\mathcal{Z} \rtimes_c R$ acts on $\mathfrak{K}$ via

$$\widetilde{\sigma}_{(z,k)}(\mathfrak{k}) := \sigma_k(\varphi) + ker(\pi_z),$$

where $\mathfrak{k} = \varphi + ker\pi_z$ lies in $\mathfrak{K}(\tilde{s}(z, k))$. Note that this action is well defined, since $\pi_z \circ \sigma_k = \pi_{z.k}$. We lift the action of $\mathcal{Z} \rtimes_c R$ on $\mathfrak{K}$ to one on

$$\mathcal{Z} * \mathfrak{K} := \{(z, \mathfrak{k}) : \mathfrak{k} \in \mathfrak{K}(z) : (z, \mathfrak{k}).(z, k) = (z.k, \tilde{\sigma}^{-1}_{(z,k)}(\mathfrak{k}))\}.$$

If the action of R on $\mathcal{Z}$ is free and proper, then the action of $\mathcal{Z} \rtimes_c R$ on $\mathcal{Z} * \mathfrak{K}$ is also free and proper. In this case, we write $\mathfrak{K}^{\mathcal{Z}}$ for the quotient space $\mathcal{Z} * \mathfrak{K}/\mathcal{Z} \rtimes_c R$. Then $\mathfrak{K}^{\mathcal{Z}}$ is naturally a bundle of elementary C*-algebras over $\mathcal{Z}/R$. In particular, $C_0(\mathcal{Z}/R, \mathfrak{K}^{\mathcal{Z}})$ has continuous trace, when R acts on $\mathcal{Z}$ freely and properly. In this case, $C^*(R, C^*(\Gamma(X)))$ is strongly Morita equivalent to $C_0(\mathcal{Z}/R, \mathfrak{K}^{\mathcal{Z}})$, where $\mathfrak{K}^{\mathcal{Z}}$ is the elementary C*-bundle over $\mathcal{Z}/R$ satisfying Fell's condition [6].

As in [12, Lemma 2.7] we can consider locally compact transformation groupoid $(\mathcal{Z}, R, \rho)$, with associated graph $\mathcal{Z} \times^{\rho}_c R$. Then $C^*(\mathcal{Z} \rtimes^{\rho}_c R) = C_0(\mathcal{Z}) \rtimes^{\rho} R$, hence $C^*(\mathcal{R}) = C^*(\Gamma(X)) \rtimes^{\rho} R$.

Proposition 2.3. *If* Γ *is a decomposable abelian groupoid, then* $C^*(\Gamma) = C^*(\mathcal{R})$*, where* $\mathcal{R} = \widehat{\Gamma(X)} \rtimes_c R$.

Proof. As in [15, page 132], we have $C^*(\mathcal{R}, \mathcal{G}) \cong C^*(\mathcal{R}, \omega)$, where ω is a 2-cocycle on $\mathcal{R}$ associated to $\mathbb{T}$-groupoid $\mathcal{G}$. By [16, Proposition 4.5] $C^*(\Gamma) \cong C^*(\mathcal{R}, \mathcal{G})$, for the decomposable abelian groupoid Γ, and $\omega = 1$, because $\mathcal{G}$ is a trivial twist on $\mathcal{R}$. □

If Γ is a decomposable abelian groupoid, then $C^*(\Gamma)$ is CCR by Theorem 2.1 and [4, Prop 4.5.3].

Corollary 2.4. *If* Γ *is a decomposable abelian groupoid, then* $primC^*(\Gamma) \cong \widehat{C^*(\Gamma)}$*, where* $primC^*(\Gamma)$ *is the set of primitive ideals of* $C^*(\Gamma)$.

Proof. By [17, Theorem 5.6.4] if A is a non-zero GCR (or CCR) C^*-algebra, then the canonical map $\hat{A} \to Prim(A)$ is a homeomorphism. □

If a C^*-algebra A is unital, $Prim(A)$ is compact but not necessarily Hausdorff [13]. Let M be a maximal ideal in the center $Z(A)$ of A. Let $I = AM$ be the ideal generated by M in A. A straightforward application of Cohen's factorization theorem shows that I is closed. Ideals of the above kind are called Glimm ideals [13].

Proposition 2.5. *If* Γ *is a decomposable abelian groupoid, then* $\widehat{C^*(\Gamma)}$ *is Hausdorff.*

Proof. Since $C^*(\Gamma) \cong C^*(\widehat{\Gamma(X)} \rtimes_c R, \mathcal{G})$, where $\mathcal{G}$ is the $\mathbb{T}$-groupoid over $\widehat{\Gamma(X)} \rtimes_c R$, $SpC^*(\Gamma) \cong SpC^*(\widehat{\Gamma(X)} \rtimes_c R, \mathcal{G})$. By [15, Proposition 3.3], $SpC^*(\widehat{\Gamma(X)} \rtimes_c R, \mathcal{G}) \cong \widehat{\Gamma(X)}/\widehat{\Gamma(X)} \rtimes_c R$, which is Hausdorff because $\widehat{\Gamma(X)}/\widehat{\Gamma(X)} \rtimes_c R \cong \bigsqcup_{[x] \in X/R}([x] \times \widehat{\Gamma(x)})$, where X/R and $\widehat{\Gamma(x)}$ are Hausdorff. □

As a result, if Γ is a decomposable abelian groupoid, then by [13, Lemma 9], every Glimm ideal in $C^*(\Gamma)$ is maximal. Indeed, for a C^*-algebra A, $Prim(A)$ is Hausdorff if and only if every Glimm ideal of A is maximal. Now $Glimm(C^*(\Gamma)) \subset \widehat{C^*(\Gamma)}$, because every maximal ideal in a C^*-algebra is primitive.

3. DUAL GROUPOID

This section is devoted to the dual groupoid of decomposable abelian groupoids. We characterize the dual groupoid and investigate properties of its C^*-algebra.

A construction of dual groupoid for a measured groupoid Γ with discrete isotropy groups such that $R(\Gamma)$ is hyperfinite (AF) is given in [22, page 1101]. Choose a family $\{\hat{\mathcal{N}}(x)\}_{x \in X}$ of groups with the following properties: first each $\hat{\mathcal{N}}(x)$ is a dense countable subgroup of $\widehat{\Gamma(x)}$; second $x \in X \mapsto \hat{\mathcal{N}}(x)$ is a Borel field, i.e. $\bigsqcup_{x\in X}\hat{\mathcal{N}}(x)$ is a Borel set in $\bigsqcup_{x\in X}\widehat{\Gamma(x)}_{x\in X}$. Then [22] saturates $\{\hat{\mathcal{N}}(x)\}$ with respect to the natural action of $R(\Gamma)$ on $\{\widehat{\Gamma(x)}\}_{x\in X}$

$$\langle\chi.k,h\rangle=\langle\chi,k.h\rangle$$

where $k.h=khk^{-1}$, thus we can assume that the field $\{\hat{\mathcal{N}}(x)\}$ is invariant under the action of $R(\Gamma)$.

Definition 3.1. The dual groupoid of an abelian groupoid Γ is constructed by $\widehat{\Gamma(X)}$ and $R(\Gamma)$ using semidirect product (defined in [1, page 80]), i.e. $\hat{\Gamma}:=\widehat{\Gamma(X)}\rtimes_s R(\Gamma)$.

Note that $\widehat{\Gamma(X)}$ is well defined by [7, prop. 2.49]. Also we have $\widehat{\widehat{\Gamma(X)}}=\Gamma(X)$ by [7, theorem 2.52]. This could be considered as a generalization of Pontryagin duality, because $\widehat{\widehat{\Gamma(X)}}\rtimes_s R\cong\Gamma$, algebraically and topologically, but to transform the Haar system, we need more conditions [18], since $R=\Gamma(X)\backslash\Gamma=\Gamma/\Gamma(X)$, set theoretically, but usually not topologically [16]. Also $R=\widehat{\Gamma(X)}\backslash\widehat{\Gamma}=\widehat{\Gamma}/\widehat{\Gamma(X)}$, set theoretically, but usually not topologically [18].

Theorem 3.2. *With the above notation,* $\frac{\Gamma(X)\rtimes_c R}{\Gamma(X)}\cong\Gamma(X)\rtimes_s R=\Gamma$. *Similarly* $\frac{\widehat{\Gamma(X)}\rtimes_c R}{\widehat{\Gamma(X)}}\cong\widehat{\Gamma(X)}\rtimes_s R=\hat{\Gamma}$.

Proof. We define an equivalence relation $\sim$ on $\Gamma(X)\rtimes_c R$ by $(k,g)\sim(t,h)$ iff $kg=th$. Since $(k,g)\in\Gamma(X)\rtimes_c R$, we have $s(g)=s(k)$, similarly $s(h)=s(t)$. If $\mathcal{N}$ is the subgroupoid induced by $\sim$, then $(k,g)(t,h)^{-1}$ and $(k,g)^{-1}(t,h)$ are in $\mathcal{N}$, and their multiplication gives (x,g) or (x,h), hence $\mathcal{N}=\Gamma(X)$. The other assertion is proved similarly. □

Remark 3.3. If Γ is a decomposable abelian groupoid, $C^*(\Gamma)\cong C^*(\Gamma(X))\rtimes R$ and $C^*(\hat{\Gamma})\cong C_0(\Gamma(X))\rtimes R$.

As in the proof of Proposition 2.4, $\widehat{C^*(\Gamma)}\cong\widehat{\Gamma(X)}/\mathcal{R}$, now we can apply [7, corollary 6.51] to deduce that $Sp[C^*(\Gamma(X))\rtimes_c R]$ is isomorphic to $\widehat{\Gamma(X)}/R$ because X/R is Hausdorff (Remark 1.3). Also we have $\widehat{C^*(\Gamma)}\cong\widehat{\Gamma(X)}/\Gamma$ when $\Gamma(X)$ has continuous Haar system, by [7, corollary 7.1]. This is because $\widehat{\Gamma(X)}/\mathcal{R}\cong\widehat{\Gamma(X)}/R\cong\widehat{\Gamma(X)}/\Gamma$, algebraically and topologically.

Finally let us show that decomposable abelian groupoids are *unimodular*, in the sense of [23, def. 3.9].

Proposition 3.4. *Decomposable abelian groupoids and their duals are unimodular.*

Proof. By [16, lemma 4.1], we can set new Haar measures $d\beta(x)$ on $\Gamma(X)$ such that $\beta^{r(\gamma)}=(Ad_\gamma)_*\beta^{s(\gamma)}$, where β is Haar system on $\Gamma(X)$ and $Ad_\gamma:\Gamma^{s(\gamma)}\to\Gamma^{r(\gamma)}$ is given by $\eta\to\gamma\eta\gamma^{-1}$, hence Γ is unimodular by [23, corollary 3.10]. We can similarly conclude that $\widehat{\Gamma}$ is unimodular. □

4. ABELIAN GROUPOIDS WITH FINITE UNIT SPACE

In this section we study the C^*-algebra of an r-discrete abelian groupoid with finite unit space.

Theorem 4.1. *Every r-discrete abelian groupoid with finite unit space is decomposable.*

Proof. Let Γ be an r-discrete abelian groupoid with finite unit space X and $R = R(\Gamma)$, then R is proper (because X is finite). Also R is an r-discrete groupoid, because by openness of $\theta = (r, s)$ [7, prop. 1.53], $\theta(X) = X$ is open in R. Finally, R has a (continuous) Haar system, because by [19, Prop. I.2.8], Γ has a base of open bisections, and for each bisection U of Γ, $\theta(U)$ is an open bisection in R, therefore R has a base of open bisections. In additions $\theta(U)$ are compact, therefore R is covered with compact open R-sets. □

Corollary 4.2. *If* Γ *is an r-discrete abelian groupoid with finite unit space, then* $\Gamma(X)$ *and* $\widehat{\Gamma(X)}$ *have Haar systems.*

Proof. The first assertion follows from the above theorem and the fact that (continuous) Haar systems on a decomposable groupoid Γ correspond bijectively to (continuous) Haar systems on the isotropy subgroupoid $\Gamma(X)$ [23, page 99]. By [16, Corollary 3.4], $\widehat{\Gamma(X)}$, equipped with the Gelfand topology, is a locally compact group bundle, and [16, Proposition 3.6] implies that if $\{\beta^x\}$ is a Haar system of $\Gamma(X)$, then $\{\widehat{\beta^x}\}$ is a Haar system for $\widehat{\Gamma(X)}$. □

We fix a Haar measure $d\beta(x)$ on $\Gamma(x)$ and a Haar measure $d\chi$ on $\widehat{\Gamma(x)}$ which are in duality, that is

$$\int\int \varphi(h)\varphi(\chi)d^{\beta(x)}(h)d\chi = \varphi(1),$$

for all $\varphi \in C_c(\Gamma(x))$.

Note that if Γ is a groupoid with finite unit space X, then $\hat{\Gamma}$ has also finite unit space X.

Corollary 4.3. *If* Γ *is an r-discrete abelian groupoid with finite unit space, then* $\widehat{\Gamma}$ *is decomposable.*

Proof. We have the exact sequences $\Gamma(X) \to \Gamma \to R(\Gamma)$ and $\hat{\Gamma}(X) \to \hat{\Gamma} \to R(\Gamma)$, in which $R(\Gamma)$ is r-discrete and proper and covered with compact open R-sets. □

When the groupoid is second countable and has continuous Haar system, if each $\Gamma(x)$ is open in Γ and R is a proper and r-discrete, then it is AF by discreteness of X in R. In this case, $ext(R, \Gamma(X)) = \{0\}$, because by [19, Prop. III.1.3(iii)], $H^2(R, \Gamma(X)) = 0$, and by [19, Prop 1.14, page 14], $ext(R, \Gamma(X)) = H^2(R, \Gamma(X))$. If such a groupoid has a compact unit space, it must have a finite unit space, because $r(\Gamma(x)) = x$ must be open in X.

In an r-discrete abelian groupoid Γ, if the principal groupoid R is proper with totally disconnected unit space X, (hence R is AF), then $\Gamma(X) \rtimes_c R$ is AF by [19, Prop. III.1.7(ii)].

Next let G be a discrete group acting on an r-discrete Hausdorff groupoid R, and let τ denote the associated action on $C^*(R)$ [8, Proposition 5.1]. Then

$$C^*(G \rtimes_s R) \cong C^*(R) \times_\tau G.$$

For a transitive groupoid Γ with isotropy group G, if Γ is r-discrete, then G is discrete and R is r-discrete, hence

$$C^*(\Gamma) \cong C^*(R) \times_\tau G.$$

This means that $C^*(\Gamma)$ is given as the crossed product of $C^*(R)$ by endomorphisms in the sense of Paschke [2, page 61].

Theorem 4.4. *If Γ is an r-discrete abelian groupoid whose principal groupoid R is AF, then $C^*(\Gamma)$ is AF.*

Proof. By [19, page 127], since $\mathbb{T}$ is a compact abelian group, $C^*(R) \rtimes \mathbb{T}$ is an AF C^*-algebra. Note that $C^*(R) \rtimes \mathbb{T}$ in Renault's notation sometimes is denoted by $C^*(R) \rtimes \hat{\mathbb{T}}$ [8]. Similarly if we use $\mathbb{Z}_n$ instead $\mathbb{T}$, since $\widehat{\mathbb{Z}_n} = \mathbb{Z}_n$, we conclude that $C^*(\mathbb{Z} \rtimes_s R)$ and $C^*(\mathbb{Z}_n \rtimes_s R)$ are AF. On the other hand, Γ is locally transitive, in the sense of Buneci, because X is totally disconnected. Now since each component of $C^*(\Gamma)$ is AF, it follows that $C^*(\Gamma)$ is also AF. □

Every AF-algebra $\mathcal{A}$ arises as the C^*-algebra of a locally finite pointed directed graph in the sense of [10].

Proposition 4.5. *Every AF-algebra (unital or not) contains a subalgebra which is diagonal.*

Proof. By [5, Theorem 1], every AF-algebra A is the C^*-algebra of a Hausdorff, r-discrete equivalence relation, which is amenable by [10, Corollary 5.5]. Thus, by [5, Theorem 1], A contains a diagonal subalgebra. □

For $A = C^*(\Gamma)$ in the above theorem, we can find an AF groupoid $\mathcal{R}$ such that $C^*(\Gamma) \cong C^*(\mathcal{R})$. We conjecture that $\mathcal{R} = \widehat{\Gamma(X)} \rtimes_c R$. By Proposition 2.3, this is true if Γ has finite unit space.

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