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Time Optimal Control for Linear Bounded Phase Coordinate Control Problems in 2-Banach Spaces

Utpalendu Adak

Joyrampur Netaji High School, Department of Mathematics,
Joyrampur, Arambagh, Hooghly, West Bengal:712413, India
utpalendu_adak@yahoo.in, utpalendu_adak@rediffmail.com

Hora Krishna Samanta

Netaji Mahavidyalaya, Department of Mathematics, Arambagh
West Bengal – 712601, India

Abstract

In this paper, time optimal control for linear bounded phase coordinate control in 2-Banach Space are considered . Necessary and sufficient conditions are deduced in 2-Banach Space**.**

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Keywords and Phrases: minimum time control, necessary and sufficient conditions, 2-Banach Spaces, supporting hyperplane, generalized 2-norm, Hahn-Banach, weak*-topology.

1. Introduction

R.N. Mukherjee([5],[6]) developed a uniform theory of time optimal control problems using functional analysis technique in Banach Space setting. Bounded phase coordinate control problem was formulated and solved by Minamide and Nakamura[4] in 1972 in Banach Space setting. R.N.Mukherjee [6] in 2002, [7] in 2007 solved a time optimal control problem for linear bounded phase coordinate control problem by functional analytic technique. Recently, the concept of 2-Banach spaces has been developed. Very authors like Mehmet Acikgoz[3] in 2007; Freese, R., Cho, Y.[1] in 2001 ; Z.Lewandowska, M.S.Molehian, A.Saadatpour[2] in 2006 have developed a uniform theory in 2-Banach space. Optimization in 2-Banach space setting is an important area of application of functional analysis. So, it may be worthwhile to make an attempt to develop an optimization theory in 2-Banach space. In this paper, we have developed the time optimal control for linear bounded phase co-ordinate control problems in 2-Banach Space. So, we consider the following problems:

Problem Statement: Let X_t, Y and Z be 2-Banach spaces, T_t be a bounded linear operators from X_t into Y and S_t be a bounded linear transformation from X_t onto Z for all $t\in(0,\infty)$. Let U_{Xt},U_Y,U_Z be the unit balls in X_t,Y,Z respectively. Let us consider the onto mapping $\widehat{T}_t: X_t\to Y\times Z$ defined by $\widehat{T}_t(u_t)=(T_tu_t,S_tu_t),u_t\in U_{Xt}\subseteq X_t$. Consider the set $C_\varepsilon(t)=\{ \widehat{T}_t(u_t)+\varepsilon U_Y\times\{0\}\},(\varepsilon>0)$.

Problem I. With S onto, $\xi \in Y$ and $\eta \in Z$, the problem is to find the possibility of reaching any given point $(\xi,\eta)\in Y\times Z$ by applying a pair$(u_t,y)\in C_\varepsilon(t)$ such that t is the minimum time taken with $\widehat{T}_t(u_t)=(\xi,\eta)$, $\eta = S_tu_t$ and $N_2\{(\xi - T_tu_t, w): \xi - T_tu_t, w\in Y\} \le \varepsilon$ $(\varepsilon>0)$.

Problem II. With T_t and S_t both into, the problem is to investigate the possibility of reaching any given point $(\xi,\eta)\in Y\times Z$ by applying a pair$(u_t,y)\in C_\varepsilon(t)$ such that t is the minimum time taken with $\widehat{T}_t(u_t)=(\xi,\eta)$, $N_2\{(\xi - T_tu_t, w): \xi - T_tu_t, w\in Y\} \le \varepsilon$ $(\varepsilon>0),u_t\in\rho U_{Xt}=\{u_t|N_1\{(u_t,v_t):u_t\in X_t\} \le\rho$, $v_t\in X_t, v_t\neq \theta$ where $C_\varepsilon(t)= \{\rho\widehat{T}_t(U_{Xt})+(\varepsilon U_Y\times U_Z)\},(\varepsilon>0)$, $t\in(0,\infty)$.

In the problems which usually arise in practice, $X_t\times Y$ is an increasing function of t in the sense that $X_{t1}\subseteq X_{t2}$, whenever $t_1\le t_2$ .Also, $\widehat{T}_{t1}$ can be considered as the restriction of $\widehat{T}_{t2}$,on X_{t1}. Thus if $t_1\le t_2$, $C_\varepsilon(t_1)\subseteq C_\varepsilon(t_2)$.

Definition of 2-Normed space 1 ([3],[8]): Let X_t be a vector space of dimension greater than one over F, where F is the real or complex number field. Suppose N_1 (.,.) be a non negative real function on $X_t\times X_t$ satisfies the following conditions:(i) $N_1(x_i,x_j)=0$ if and only if x_i and x_j are linearly dependent vectors.(ii)$N_1(x_i,x_j)= N_1 (x_j, x_i)$ for all $x_i,x_j \in X_t$. (iii) $N_1 (\lambda x_i,x_j)=|\lambda| N_1 (x_i,x_j)$ for all $\lambda\in F$ and for all $x_i,x_j\in X_t$. (iv) $N_1 (x_i+x_j,z)\le N_1 (x_i,z)+ N_1 (x_j,z)$ for all $x_i,x_j,z \in X_t$. Then N_1 (.,.) is called a 2-norm on X_t and (X_t, N_1 (.,.)) is called a linear 2-normed space. Also if X_t and Y are 2-Banach spaces over the field of real numbers, it can be verified that $X_t\times Y$ is also 2-Banach space with respect to the 2-norm N_5 (.,.) where

$N_5 \{(x_i,y_i),(x_j,y_j)\} = \min\{N_1(x_i,x_j),N_2(y_i,y_j)\}$, i.e. $N_5(.,.) = \min\{N_1(.,.),N_2(.,.)\}$,

$N_1(.,.)$ & $N_2(.,.)$ are 2-norms in X_t & Y respectively and $N_5 \{(x_i,y_i),(x_j,y_j)\}=0$ iff either x_i,x_j are linearly dependent in X_t or y_i,y_j are linearly dependent in Y. Let $N_3,N_4,N_1',N_2',N_3',N_4'$ are the 2-norms of the spaces $Y\times Z,Z,X_t',Y',(Y\times Z)',Z'$ respectively. Where $N_3' (.,.) = \min\{N_2' (.,.),N_4' (.,.)\}$ and X_t' denotes the conjugate of X_t. Let $\phi_1:Y\to R$ & $\phi_2: Z\to R$ be two functionals. Then $\phi_1\in Y'$, $\phi_2\in Z'$, $f_t\in X_t'$;$(\phi_1,\phi_2)\in (Y\times Z)'$, where $f_t: X_t\to R$ be a functional.

Definition 2: Let $U_{Xt}=\{x_t:N_1(\alpha,x_t)\le 1,x_t\in X_t\},\alpha\in X_t, \alpha\neq\theta;U_Y=\{y:N_2(\beta,y) \le 1,y\in Y\},\beta\in Y$, $\beta\neq\theta$ and $U_Z=\{z:N_4(\gamma,z) \le 1,z\in Z\},\gamma\in Z$, $\gamma\neq\theta$ be the unit balls in X_t,Y and Z respectively.

Definition 3: A 2-Banach space B is called smooth or rotund according as its unit ball i.e.,$B_{(\alpha,\beta)} =\{(x,y):N_5\{(\alpha,\beta),(x,y)\}\leq 1,(x,y)\in X_t\times Y\},(\alpha,\beta)\in X_t\times Y$,is smooth or rotund. A convex body K in B called rotund (or strictly convex)if K contains no straight-line segments in its boundary. A 2-Banach space B is called smooth if at each of its boundary points, there is a unique hyperplane of support of K.

Definition 4: A control $u_t\in U_{Xt}$ will be called an admissible control if $N_1\{(u_t,v_t):u_t,v_t\in X_t\}\leq 1$.

Definition 5: The set of all points $(\xi,\eta)\in Y\times Z$ such $\widehat{T}_t(u_t)= (\xi,\eta),\eta= S_t u_t$, $N_2\{(\xi - T_t u_t,w):w\in Y\} \leq\varepsilon$ for all $u_t\in U_{Xt}\subseteq X_t$ will be called the Reachable set w.r.to the linear transformation $\widehat{T}_t$ and will be denoted by $C_\varepsilon(t)$.

Definition 6: A pair $(\xi,\eta)\in Y\times Z$, is called regular if there exists at least one element $u_t\in U_{Xt}\subseteq X_t$ satisfying $\eta= S_t u_t$, $N_2\{(\xi - T_t u_t,w):w\in Y\} \leq\varepsilon$.

Definition 7[2]: Let X and Y be real linear spaces. Denote by D a non-empty subset of X × Y such that for every $x \in X$, $y \in Y$ the sets $D_x = \{y \in Y;(x,y) \in D\}$ and $D^y = \{x \in X;(x,y) \in D\}$ are linear subspaces of the spaces Y and X, respectively.
A function $N_5(.,.) : D \to (0, \infty)$ will be called a generalized 2-norm on D if it satisfies the following condition:
(i) $N_5(x, \alpha y) = |\alpha| N_5(x, y) = N_5(\alpha x, y)$ for any real number α and all $(x ,y) \in D$;(ii) $N_5(x, y + z) \leq N_5(x, y) + N_5(x, z)$ for $x \in X$, $y, z \in Y$ with (x, y), $(x, z) \in D$;(iii) $N_5(x + y, z) \leq N_5(x, z) + N_5(y, z)$ for $x,y \in X$, $z \in Y$ with (x, z), $(y, z) \in D$. The D is called a 2-normed set. In particular, if $D = X \times Y$, the function $N_5(.,.)$ is said to be a generalized 2-norm on X × Y and the pair $(X \times Y, N_5(.,.))$ is called a generalized 2-norm space. If X = Y, then the generalized 2-normed space $(X \times X, N_1(.,.))$ is denoted by $(X, N_1(.,.))$. In the case that X = Y, $D = D^{-1}$, where $D^{-1} = \{(y, x) : (x, y) \in D\}$, and $N_5(x, y) = N_5(y, x)$ for all $(x, y) \in D$, we call $N_5(.,.)$ a generalized symmetric 2-norm and D a symmetric 2-norm set. Also, let $(X, N_1(.))$ be a normed space. Then $N_1(x, y) = N_1(x). N_1(y)$, for all $x, y \in X$, is a 2-norm on X × X. So, $(X, N_1(.,.))$ is a generalized 2-normed space. Generalized 2-normed space is a 2-normed space. Throughout the paper $N_1,N_2,N_3,N_4,N_5,N_1',N_2',N_3',N_4',N_5'$ denote the 2-norms of the spaces X_t,Y,Y×Z,Z, $X_t\times Y$, X_t', Y′,(Y×Z)′,Z′,$(X_t\times Y)'$ respectively which are defined earlier in definition 1.

Proposition 1. The set $C_\varepsilon(t)$ is a convex body.

Proposition 2. Let $(\xi,\eta) \in \partial C_\varepsilon(t)$ be a regular pair and (ϕ_1, ϕ_2) $(\neq(0,0))\in(Y\times Z)'$ be a supporting hyperplane of $C_\varepsilon(t)$ at (ξ,η).Then (i) $\langle(\xi,\eta),(\phi_1, \phi_2)\rangle \geq \alpha N_1'\{(T'_t\phi_1 +S'_t\phi_2,f_t) : f_t\in X_t'\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$ and (ii) $N_1'\{(T'_t\phi_1 +S'_t\phi_2,f_t) : f_t\in X_t'\} \neq 0$ when T'_t, S'_t are conjugate to T_t and S_t, respectively.

Proof: Since $C_\xi(t)$ is a convex body, by Hahn-Banach theorem[8], there exists a hyperplane $(\phi_1, \phi_2) \in (Y \times Z)'$ such that $\langle(\xi, \eta),(\phi_1, \phi_2)\rangle \geq \langle(T_t u_t, S_t u_t) + \varepsilon(y, 0), (\phi_1, \phi_2)\rangle$, $\forall u_t \in U_{xt}$, $y \in U_y$. Hence taking supremum on the right side, we get (i). To prove (ii) if possible let $T_t'\phi_1 + S'_t\phi_2 = 0$. Then evidently we have $\phi_1 \neq 0$ and $u_t \in S_t^{-1}(\eta) = \{u_t : \eta = S_t u_t \in U_{xt} \subset X_t\}$.

Now, $N_2\{(\xi - T_t u_t, w): w \in Y\} \geq \langle \xi - T_t u_t, \phi_1\rangle = \langle \xi, \phi_1\rangle + \langle u_t, S'_t\phi_2\rangle$

$= \langle(\xi,\eta),(\phi_1, \phi_2)\rangle \geq \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$ ……(by (i)).

Hence $N_2\{(\xi - T_t u_t, w): w \in Y\} \geq \varepsilon$, $\forall u_t \in S_t^{-1}(\eta)$ which contradicts the rerularity of the pair (ξ, η).

Corollary 1: The set $C_\varepsilon(t)$ is closed in $Y \times Z$ if either of the following conditions hold:

(i) X_t is a reflexive 2-Banach Space,

(ii)(a) There exist 2-normed linear spaces whose conjugates are X_t,Yand Z respectively,

(b) There exist bounded linear transformations whose conjugates are T_t and S_t respectively.

Auxiliary Problem: Let $(\xi,\eta) \in \partial C_\varepsilon(t)$ with $\eta = S_t u_t$, $N_2\{(\xi - T_t u_t, w): w \in Y\} \leq \varepsilon$ for some given time t. Then determine $u_t \in U_{Xt} \subset X_t$,such that $\widehat{T}_t(u_t) = (\xi,\eta)$ and $N_1\{(u_t,v_t): u_t,v_t \in X_t\}$is minimum, where $\partial C_\varepsilon(t)$ denotes the boundary of $C_\varepsilon(t)$. Such an u_t will be called an optimal control and t is the corresponding minimum time to reach (ξ,η).

Proposition 3 : An admissible control which will be optimal in the above sense must satisfy $N_1\{(u_t,v_t): u_t,v_t \in X_t\} = 1$.

Proof : Let (ϕ_1,ϕ_2) be a supporting hyperplane to a regular pair $(\xi,\eta) \in \partial C_\varepsilon(t)$. Then for all $u_t \in S_t^{-1}(\eta)$, we have

$N_1\{(u_t,v_t): u_t,v_t \in X_t\}\ N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t): f_t \in X'_t\} + N_2\{(\xi - T_t u_t, w): w \in Y\} N'_2\{(\phi_1, \phi_3): \phi_1,\phi_3 \in Y'\}$

$\geq \langle u_t, T'_t\phi_1 + S'_t\phi_2\rangle + \langle \xi - T'_t u_t, \phi_1\rangle$

$= \langle(\xi,\eta),(\phi_1, \phi_2)\rangle \geq N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t) : f_t \in X'_t\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$.

Hence $(N_1\{(u_t,v_t): u_t,v_t \in X_t\} - 1)\ N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t) : f_t \in X'_t\} \geq (\varepsilon - N_2\{(\xi - T_t u_t, w): w \in Y\})\ N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$, $\forall\ u_t \in S_t^{-1}(\eta)$. Since $N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t) : f_t \in X'_t\} \neq 0$, therefore $N_1\{(u_t,v_t): u_t,v_t \in X_t\} \geq 1$. Consequently $N_1\{(u_t,v_t): u_t,v_t \in X_t\} = 1$.

Proposition 4 : Let $(\xi,\eta) \in \partial C_\varepsilon(t)$ be a regular pair and $(\phi_1,\phi_2) \neq (0,0) \in (Y \times Z)'$ be a supporting hyperplane at (ξ,η). Then

$\langle(\xi,\eta),(\phi_1, \phi_2)\rangle = N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t) : f_t \in X'_t\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$.

Proof : By Proposition 2, we have$\langle(\xi,\eta),(\phi_1, \phi_2)\rangle \geq N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t) : f_t \in X'_t\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$. Let $u_{t0} \in U_{xt} \subseteq X_t$ be the optimal control to reach (ξ,η). Then

$\langle(\xi,\eta),(\phi_1, \phi_2)\rangle = \langle \xi - T'_t u_t, \phi_1\rangle + \langle u_t, T'_t\phi_1 + S'_t\phi_2\rangle$

$\leq N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t) : f_t \in X'_t\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$.

Hence $\langle(\xi,\eta),(\phi_1, \phi_2)\rangle = N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t): f_t \in X'_t\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$.

Proposition 5: If $\langle(\xi,\eta),(\phi_1, \phi_2)\rangle = N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t) : f_t \in X'_t\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$ for some $(\xi,\eta) \in C_\varepsilon(t)$ and $(\phi_1,\phi_2) \neq (0,0) \in (Y\times Z)'$, then $(\xi,\eta) \in \partial C_\varepsilon(t)$.(Assuming the conditions (i) and (ii) of the corollary 1 and that (ϕ_1,ϕ_2) defines a supporting hyperplane to $C_\varepsilon(t)$ at (ξ,η)).

Proof : By the assumptions one can show that $C_\varepsilon(t)$ weakly compact (or weak* compact) in some appropriate topology. Also, $C_\varepsilon(t)$ is a convex body. Since $(\phi_1,\phi_2) \in (Y\times Z)'$ hence there exists a point $(\xi_1,\eta_1) \in \delta C_\varepsilon(t)$ such that (ϕ_1,ϕ_2) defines a supporting hyperplane to $C_\varepsilon(t)$ at (ξ_1,η_1). Therefore $\langle(\xi_1,\eta_1),(\phi_1, \phi_2)\rangle = N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t) : f_t \in X'_t\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$. But by hypothesis $\langle(\xi,\eta),(\phi_1, \phi_2)\rangle = N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t) : f_t \in X'_t\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$ for some $(\xi,\eta) \in C_\varepsilon(t)$. Now, if $(\xi,\eta) \notin \delta C_\varepsilon(t)$, then $\langle(\xi,\eta),(\phi_1, \phi_2)\rangle < \langle(\xi_1,\eta_1),(\phi_1, \phi_2)\rangle$, i.e., $\langle(\xi,\eta),(\phi_1, \phi_2)\rangle < N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t): f_t \in X'_t\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$, which contradicts the hypothesis.

Hence $(\xi,\eta) \in \delta C_\varepsilon(t)$. Also, since $\langle(\xi_2,\eta_2),(\phi_1, \phi_2)\rangle \le \langle(\xi_1,\eta_1),(\phi_1, \phi_2)\rangle = \langle(\xi,\eta),(\phi_1, \phi_2)\rangle$ for some $(\xi_2,\eta_2) \in \delta C_\varepsilon(t)$, consequently (ϕ_1, ϕ_2) defines a supporting hyperplane to $C_\varepsilon(t)$ at (ξ,η).

Corollary 2: Let $(\xi,\eta) \in \partial C_\varepsilon(t)$ be a regular pair when t is given terminal time and $(\phi_1,\phi_2) \in (Y\times Z)'$ be a supporting hyperplane at (ξ,η). Let $u_{t0} \in U_{Xt} \subset X_t$ be the optimal control such that $\widehat{T}_t(u_{t0}) = (\xi,\eta), \eta = S_t u_{t0}$, $N_2\{(\xi - T_t u_{t0}, w) : w \in Y\} \le \varepsilon$. Then (u_{t0},y) maximizes $\langle(u_t,y), (T'_t\phi_1 + S'_t\phi_2, \phi_1)\rangle$.

Proof : Since (ϕ_1,ϕ_2) is the supporting hyperplane at (ξ,η). Hence $\langle(\xi_1,\eta_1),(\phi_1, \phi_2)\rangle \le \langle(\xi,\eta),(\phi_1, \phi_2)\rangle, \forall (\xi_1,\eta_1) \in C_\varepsilon(t)$,

i.e., $\langle \xi_1 - T'_t u_t, \phi_1\rangle + \langle u_t, T'_t\phi_1 + S'_t\phi_2\rangle \le \langle \xi - T'_t u_t, \phi_1\rangle + \langle u_t, T'_t\phi_1 + S'_t\phi_2\rangle$,
i.e., $\langle(u_t, \varepsilon y),(T'_t\phi_1 + S'_t\phi_2, \phi_1) \le \langle(u_{t0}, \varepsilon y_0),(T'_{t0}\phi_1 + S'_{t0}\phi_2, \phi_1)$. This proves the corollary.

Let $u_{t0} (\neq 0)$ be the optimal solution. Hence $N_2\{(\xi - T_t u_t, w): w \in Y\} \le \varepsilon$, $\eta = S_{t0} u_{t0}$. Also, it follows that $(\xi,\eta) \in \partial C_\varepsilon(t_0)$, where $N_1\{(u_{t0}, v_{t0}): u_{t0}, v_{t0} \in X_{t0}\} = 1$. Let (ϕ_1, ϕ_2) be a hyperplane of support to $C_\varepsilon(t_0)$ at (ξ,η). Then, by Proposition 2, $T'_{t0}\phi_1 + S'_{t0}\phi_2 \neq 0$ and $\langle(\xi_1,\eta_1),(\phi_1, \phi_2)\rangle \ge$ $N_1'\{(T'_{t0}\phi_1 + S'_{t0}\phi_2, f_{t0}) : f_{t0} \in X'_{t0}\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$.

On the other hand, we have
$\langle(\xi_1,\eta_1),(\phi_1, \phi_2)\rangle = \langle \xi - T'_t u_t, \phi_1\rangle + \langle u_{t0}, T'_{t0}\phi_1 + S'_{t0}\phi_2\rangle$
$\le N_1'\{(T'_{t0}\phi_1 + S'_{t0}\phi_2, f_{t0}): f_{t0} \in X'_{t0}\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$.

Hence we conclude

$$u_{t0} = \overline{(T'_{t0}\phi_1 + S'_{t0}\phi_2)},\quad \xi - T'_{t0}u_{t0} = \varepsilon\,\overline{\phi_1} \quad \text{and} \quad \eta = S_{t0}u_{t0} = S_{t0}\,\overline{(T'_{t0}\phi_1 + S'_{t0}\phi_2)}.$$

Note 1: It should be noted that, if $(\xi,\eta) \in \partial C_\varepsilon(t)$, then the time optimal control to derive the system from origin to (ξ,η) will be given $u_{t0} = \overline{T'_{t0}\phi_1 + S'_{t0}\phi_2}$, when (ϕ_1,ϕ_2) defines the supporting hyperplane to $\partial C_\varepsilon(t)$ at (ξ,η). Thus the solution of the auxiliary problem leads to the solution of the time optimal control problem. The above procedure of solving the time optimal control problem will evidently be valid so long as $C_\varepsilon(t)$ is an increasing function of t in the sense $C_\varepsilon(t_1) \subset C_\varepsilon(t_2)$ whenever $t_1<t_2$ and $\partial C_\varepsilon(t_1) \cap \partial C_\varepsilon(t_2)=\varnothing$ and assume global controllability, i.e., given any point $(\xi,\eta)\in Y\times Z$, there exists a t such that $(\xi,\eta)\in\partial C_\varepsilon(t)$, for $\varepsilon >0$. We shall prove a necessary and sufficient condition for $C_\varepsilon(t)$ to be a strictly increasing function of t.

Proposition 6: The point $(\xi,\eta)\in C_\varepsilon(t)$ will be in $\partial C_\varepsilon(t)$ at the time $t=t_f$ iff

$$\max_{(\phi_1,\phi_2)\neq(0,0)\in(Y\times Z)'} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t) : f_t\in X'_t\} + \varepsilon N'_2\{(\phi_1,\phi_3) : \phi_1,\phi_3 \in Y'\}} = 1$$

(with the hypothesis (i) and(ii) of Proposition 2 and corollary 1).

Sufficiency. Suppose

$$\max_{(\phi_1,\phi_2)\neq(0,0)\in(Y\times Z)'} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_t\phi_1 + S'_t\phi_2, f_t) : f_t\in X'_t\} + \varepsilon N'_2\{(\phi_1,\phi_3) : \phi_1,\phi_3 \in Y'\}} = 1.$$

Let the minimum be attained for some $\phi = (\phi_\xi, \phi_\eta) \in (Y\times Z)'$. Then

$\langle(\xi,\eta),(\phi_1, \phi_2)\rangle \geq N_1'\{(T'_{tf}\phi_\xi + S'_{tf}\phi_\eta, f_{tf}) : f_{tf}\in X'_{tf}\} + \varepsilon N'_2\{(\phi_{\xi1},\phi_{\xi3}) : \phi_{\xi1},\phi_{\xi3} \in Y'\}$.

Hence by Proposition 5, $(\xi,\eta)\in\partial C_\varepsilon(t_f)$ and (ϕ_ξ, ϕ_η) define a supporting hyperplane to $C_\varepsilon(t_f)$ at (ξ,η).

Necessity. Let $(\xi,\eta)\in\partial C_\varepsilon(t_f)$. Then $\langle(\xi,\eta), (\phi_\xi, \phi_\eta)\rangle = N_1'\{(T'_{tf}\phi_\xi + S'_{tf}\phi_\eta, f_{tf}) : f_{tf} \in X'_{tf}\} + \varepsilon N'_2\{(\phi_{\xi1},\phi_{\xi3}) : \phi_{\xi1},\phi_{\xi3} \in Y'\}$, where (ϕ_ξ, ϕ_η) is the supporting hyperplane to $C_\varepsilon(t_f)$ at (ξ,η).

$$\frac{\langle(\xi,\eta),(\phi_\xi,\phi_\eta)\rangle}{N_1'\{(T'_{tf}\phi_\xi + S'_{tf}\phi_\eta, f_{tf}) : f_{tf}\in X'_{tf}\} + \varepsilon N'_2\{(\phi_{\xi1},\phi_{\xi3}) : \phi_{\xi1},\phi_{\xi3} \in Y'\}} = 1. \quad ..(A)$$

We want to show that (A) gives the maximum value of L.H.S. for all $\phi = (\phi_1, \phi_2) \in (Y\times Z)'$.

Let $\psi=(\psi_1,\psi_2)$ be any other element $\in Y^*\times Z^*$. If ψ is a supporting hyper plane to $C_\varepsilon(t_f)$ at (ξ,η), then (A) holds. So let us assumed that $\psi\in Y^*\times Z^*$ is not a supporting hyper plane to $C_\varepsilon(t_f)$ at (ξ,η).By [8] it can be shown that $C_\varepsilon(t_f)$ is convex, weakly compact, closed and bounded set.

Consequently by Proposition 2 and corollary 1, corresponding to $\psi=(\psi_1,\psi_2)\in Y^*\times Z^*$, there exists a point $(\xi_1,\eta_1)\in C_\varepsilon(t_f)\cap\partial C_\varepsilon(t_f)$ such that ψ is the supporting hyper plane at (ξ_1,η_1).

Hence we have $\langle(\xi,\eta),(\psi_1,\psi_2)\rangle \leq \langle(\xi_1,\eta_1),(\psi_1,\psi_2)\rangle = N_1'\{(T'_{tf}\psi_1 + S'_{tf}\psi_2, f_{tf}) : f_{tf}\in X'_{tf}\} + \varepsilon N'_2\{(\psi_1,\psi_3):\psi_1,\psi_3\in Y'\}$ and $\dfrac{\langle(\xi,\eta),(\psi_1,\psi_2)\rangle}{N_1'\{(T'_{tf}\psi_1 + S'_{tf}\psi_2, f_{tf}) : f_{tf}\in X'_{tf}\} + \varepsilon N'_2\{(\psi_1,\psi_3) : \psi_1,\psi_3 \in Y'\}}$

≤ 1. This proves

$$\operatorname{Max}_{\psi=(\psi_1,\psi_2)} \frac{\langle(\xi,\eta),(\psi_1,\psi_2)\rangle}{N_1'\{(T'_{tf}\psi_1 + S'_{tf}\psi_2, f_{tf}) : f_{tf}\in X'_{tf}\} + \varepsilon N'_2\{(\psi_1,\psi_3) : \psi_1,\psi_3 \in Y'\}} = 1.$$

Proposition 7: Let $(\xi,\eta)\in C_\varepsilon(t_f)\cap\partial C_\varepsilon(t_f)$. Then

$$\max_{\phi=(\phi_1,\phi_2)\in(Y\times Z)'} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_t\phi_1+S'_t\phi_2,f_t) : f_t\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}} = \leq \text{ or } \geq 1$$

according as $t \geq t_f$ or $t \leq t_f$. Moreover,the maximum is attained at a point $\phi=(\phi_\xi,\phi_\eta)\in(Y\times Z)'$, where ϕ defines a supporting hyperplane to $\partial C_\varepsilon(t)$ at the intersection with the ray through (ξ,η).To prove the above property we require the following Lemma.

Lemma1: Let $(\xi,\eta)\in\partial C_\varepsilon(t_1)$, $t_2>t_1$. Then the ray $(k\xi,k\eta)$, $k>0$ intersects $\partial C_\varepsilon(t_2)$ at some point $(\xi_1,\eta_1)= (l\xi,l\eta)$, $l\geq 0$ above $C_\varepsilon(t_1) \subset C_\varepsilon(t_2)$ whenever $t_1<t_2$.
Proof: Since $C_\varepsilon(t_2)$ is bounded, there will exist a $k>0$, say $k=k_0$, such that $(k_0\xi,k_0\eta)\in C_\varepsilon(t_2)$.
Consider the portion of the ray $R=[(k\xi,k\eta),0\leq k\leq k_0]$. We now consider a set S defined by $S=\{k : k(\xi,\eta)\in C_{\alpha 2}(t_2)\}$. Let $l=\sup_{k\in S} k$. Evidently supremum exists $(k\leq k_0)$.Consequently $l\geq 1$.

Now there exist a sequence $\{k_n\}$ such that $\lim k_n=l$ and $k_n(\xi,\eta)=(k_n\xi,k_n\eta)=(X_n, y_n)\in R\cap C_\varepsilon(t_2)$. Again, since R is compact, there is a subsequence $\{x_{nk}\}$ such that $\lim (x_{nk}, y_{nk})=(x_0 ,y_0)\in R$. Also as $\{x_{\eta k},y_{nk}\}\in C_\varepsilon(t_2)$ and $C_\varepsilon(t_2)$ is closed, there for $(x_0,y_0)=l(\xi,\eta)\in C_\varepsilon(t_2)$.
Now $(x_0,y_0)\notin \text{Int } C_\varepsilon(t_2)$. Because, if $(x_0,y_0)\in \text{Int } C_\varepsilon(t_2)$ then there will be an open sphere s_δ of radius δ which will be contained entirely within $C_\varepsilon(t_2)$. Consider the point $(x,y)= (x_0,y_0)+(\delta/2)\{(\xi,\eta)/ N_3(\xi,\eta)\}$ Then $(x,y)\in s_\delta$, and $(x,y)\in R$. But then $(x,y)=\{1+\delta/2N_3(\xi,\eta)\} (\xi,\eta)$, which contradicts the fact that $l=\sup_{k\in S} k$. This completes the proof of the lemma.

Proof of Proposition 7: We shall prove the theorem for $t_2\geq t_1$, we shall show

$$\max_{\psi=(\psi_1,\psi_2)} \frac{\langle(\xi,\eta),(\psi_1,\psi_2)\rangle}{N_1'\{(T'_{t2}\psi_1 +S'_{t2}\psi_2,f_{t2}) : f_{t2}\in X'_{t2}\} + \varepsilon N'_2\{(\psi_1, \psi_3): \psi_1, \psi_3 \in Y'\}} \leqslant 1$$

for a given $(\xi,\eta)\in C_\varepsilon(t_1)\cap\partial C_\varepsilon(t_1)$. Now, as $t_2\geq t_1$, we have $X_{t1}\subseteq X_{t2}$ and $N_1'\{(T'_{t1}\psi_1 + S'_t\psi_2 ,f_{t1}) : f_{t1}\in X'_{t1}\} + \varepsilon N'_2\{(\psi_1, \psi_3):\psi_1, \psi_3 \in Y'\} < N_1'\{(T'_{t2}\psi_1 +S'_{t2}\psi_2,f_{t2}):f_{t2}\in X'_{t2}\} + \varepsilon N'_2\{(\psi_1, \psi_3):\psi_1,\psi_3 \in Y'\}$ (by assumption).The transformation $\bar{T}_{t2}$ is such that $\bar{T}_{t1}$ is the restriction of $\bar{T}_{t2}$ on U_{t1}.Hence $C_\varepsilon(t_1) = \bar{T}_{t1}(U_{t1}) = \bar{T}_{t2}(U_{t1}) \subset \bar{T}_{t2}(U_{t2}) = C_\varepsilon(t_2)$. Thus $(\xi,\eta)\in C_{\alpha 2}(t_2)$. Let $\psi=(\psi_1,\psi_2) \in(Y\times Z)^*= Y^*\times Z^*$, where $(Y\times Z)^*$ is the conjugate space to $Y\times Z$. Consequently there exists a point $(\xi',\eta')\in\partial C_\varepsilon(t_2)$, such that $\psi=(\psi_1,\psi_2)$ defines a supporting hyper plane to $C_\varepsilon(t_2)$ at (ξ',η').Again, since $(\xi',\eta')\in\partial C_\varepsilon(t_2)$ and $\psi=(\psi_1,\psi_2)$ define a supporting hyperplane to $C_\varepsilon(t_2)$ at (ξ',η'), hence we can write $\langle(\xi' ,\eta '),(\psi_1,\psi_2)\rangle=N_1'\{(T'_{t2}\psi_1 +S'_{t2}\psi_2,f_{t2}) : f_{t2}\in X'_{t2}\} + \varepsilon N'_2\{(\psi_1, \psi_3): \psi_1, \psi_3 \in Y'\}$ and $\langle(\xi,\eta),(\psi_1,\psi_2)\rangle\leq\langle(\xi ',\eta'),(\psi_1,\psi_2)\rangle$ .Hence

$$\frac{\langle(\xi,\eta),(\psi_1,\psi_2)\rangle}{N_1'\{(T'_{t2}\psi_1 +S'_{t2}\psi_2,f_{t2}) : f_{t2}\in X'_{t2}\} + \varepsilon N'_2\{(\psi_1, \psi_3): \psi_1, \psi_3 \in Y'\}} \leqslant 1$$

$$\sup_{\psi=(\psi_1,\psi_2)\in(Y\times Z)^*} \frac{\langle(\xi,\eta),(\psi_1,\psi_2)\rangle}{N_1'\{(T'_{t2}\psi_1+S'_{t2}\psi_2,f_{t2}) : f_{t2}\in X'_{t2}\}+\varepsilon N'_2\{(\psi_1,\psi_3): \psi_1,\psi_3\in Y'\}} \leqslant 1$$

Now, $(\xi,\eta)\in\partial C_\varepsilon(t_1)$ and let $(\xi_\psi,\eta_\psi)=l(\xi,\eta)\in\partial C_\varepsilon(t_2)$, $t_2\geq t_1$ (by the above Lemma), where $\psi=(\psi_1,\psi_2)$ define the supporting hyperplane to $\partial C_\varepsilon(t_2)$ at (ξ_ψ,η_ψ). Hence

$$\frac{\langle(\xi_\psi,\eta_\psi),(\psi_1,\psi_2)\rangle}{N_1'\{(T'_{t2}\psi_1+S'_{t2}\psi_2,f_{t2}) : f_{t2}\in X'_{t2}\}+\varepsilon N'_2\{(\psi_1,\psi_3): \psi_1,\psi_3\in Y'\}} = 1$$

Therefore

$$\frac{\langle(\xi_\psi,\eta_\psi),(\psi_1,\psi_2)\rangle}{N_1'\{(T'_{t2}\psi_1+S'_{t2}\psi_2,f_{t2}) : f_{t2}\in X'_{t2}\}+\varepsilon N'_2\{(\psi_1,\psi_3): \psi_1,\psi_3\in Y'\}} \leqslant 1$$

Consequently, supremum is attained at a point $\phi=(\phi_1,\phi_2)=\ \psi=(\psi_1,\psi_2)\in(Y\times Z)^*$. Thus we have proved the theorem for $t_2\geq t_1$. Similarly, we can show that if $t_2\leq t_1$,

$$\sup_{\psi=(\psi_1,\psi_2)} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_{t2}\psi_1+S'_{t2}\psi_2,f_{t2}):f_{t2}\in X'_{t2}\}+\varepsilon N'_2\{(\psi_1,\psi_3):\psi_1,\psi_3\in Y'\}} \leqslant 1$$

This completes the proof of Proposition.

Proposition 8. Let $t_1<t_2$ and to $\widehat{T}_{t1}: X_{t1}\rightarrow Y\times Z$, $\widehat{T}_{t2}: X_{t2}\rightarrow Y\times Z$ be bounded linear transformations. Then $C_\varepsilon(t_1)\subseteq C_\varepsilon(t_2)$ and $\partial C_\varepsilon(t_1)\cap\partial C_\varepsilon(t_2)=\varnothing$, iff $N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}>N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3): \phi_1,\phi_3\in Y'\}$, where $\phi=(\phi_1,\phi_2)(\neq(0,0))\in(Y\times Z)'$.

Proof: We have already assumed that if $t_1<t_2$ then $X_{t1}\subset X_{t2}$. Let $C_\varepsilon(t_1)$ and $C_\varepsilon(t_2)$ be reachable region in $Y\times Z$ with respect to the transformations $\overline{T}_{t1}$ and $\overline{T}_{t2}$ corresponding to the time $t_1$and t_2, respectively. It has been shown earlier that and $\overline{T}_{t2}$ are linear and onto, $\overline{T}_{t1}$ can be regarded as the restriction of $\overline{T}_{t2}$ over $X_{t1}\times Y$. We have $C_\varepsilon(t_1)=\ \overline{T}_{t1}(U_{Xt1})$ and $C_{\alpha2}(t_2)=\ \overline{T}_{t2}(U_{Xt2})$ Let $(\xi,\eta)\in C_{\alpha1}(t_1)$. Then as $u_{t1}\in Ux_{t1}\subset Ux_{t2}$, hence $\overline{T}_{t2}(u_{t1})\in C_\varepsilon(t_2)$. But $\overline{T}_{t1}(u_{t1})=\ \overline{T}_{t2}(u_{t1})$, since $\overline{T}_{t1}$ is the restriction of $\overline{T}_{t2}$ on $Ux_{t1}\subset X_{t1}$. Therefore $(\xi,\eta)\in C_\varepsilon(t_2)$, $C_\varepsilon(t_1)\subseteq C_\varepsilon(t_2)$.To prove the next part, let us assume $N_1'\{(T'_t\psi_1+S'_{t1}\psi_2,f_{t2}):f_{t2}\in X'_{t2}\}+\varepsilon N'_2\{(\psi_1,\psi_3):\psi_1,\psi_3\in Y'\} < N_1'\{(T'_{t2}\psi_1+S'_{t2}\psi_2,f_{t2}):f_{t2}\in X'_{t2}\}+\varepsilon N'_2\{(\psi_1,\psi_3):\psi_1,\psi_3\in Y'\}$.We shall show that $\partial C_\varepsilon(t_1)\cap\partial C_\varepsilon(t_2)=\varnothing$. If possible, let $\partial C_\varepsilon(t_1)\cap\partial C_\varepsilon(t_2)\neq\varnothing$. Let $\phi=(\phi_1,\phi_2)\in Y^*\times Z^*$. Then corresponding to ϕ we can find, by Proposition 4, a point $(\xi_1,\eta_1)\in\partial C_\varepsilon(t_1)$ and point $(\xi_2,\eta_2)\in\partial C_\varepsilon(t_2)$ such that $\phi=(\phi_1,\phi_2)$ is the supporting hyper plane at $(\xi_1,\eta_1)\in\partial C_\varepsilon(t_1)$ and at $(\xi_2,\eta_2)\in\partial C_\varepsilon(t_2)$. Also, $\langle(\xi_1,\eta_1),(\phi_1,\phi_2)\rangle=N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3): \phi_1,\phi_3\in Y'\}$, $\langle(\xi_2,\eta_2),(\phi_1,\phi_2)\rangle = N_1'\{(T'_2\phi_1+S'_{t2}\phi_2,f_{t2}): f_{t2}\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3): \phi_1,\phi_3\in Y'\}$. Again by a previous lemma, corresponding to $(\xi_2,\eta_2)\in\partial(C_\varepsilon t_2)$, we can find a point $(\xi',\eta')\in\partial C_\varepsilon(t_1)$ such that $(\xi',\eta')=l(\xi_2,\eta_2)$ where $l\leq1$. Now we have

$N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}>N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1,$

ϕ_3): ϕ_1, $\phi_3 \in Y'\}$, (by hypothesis). Hence $\langle(\xi_2,\eta_2),(\phi_1,\phi_2)\rangle > \langle(\xi_1,\eta_1),(\phi_1,\phi_2)\rangle$. Also, $\langle(\xi_1,\eta_1),(\phi_1,\phi_2)\rangle > \langle(\xi',\eta'),(\phi_1,\phi_2)\rangle$, Since $\phi=(\phi_1,\phi_2)$ is the supporting hyper plane to (ξ_1,η_1) and (ξ',η') is any point in $C_\varepsilon(t_1)$. Consequently $\langle(\xi_2,\eta_2),(\phi_1,\phi_2)\rangle>\langle(\xi',\eta'),(\phi_1,\phi_2)\rangle$,i.e.$\langle(\xi_2,\eta_2),(\phi_1,\phi_2)\rangle> \langle l(\xi_2,\eta_2),(\phi_1,\phi_2)\rangle=l\langle(\xi_2,\eta_2),(\phi_1,\phi_2)\rangle$. Since $((\xi',\eta')=l(\xi_2,\eta_2)$, $l\leq1$).Hence $\partial C_\varepsilon(t_1)\cap \partial C_\varepsilon(t_2)= \varnothing$.

Conversely, let $\partial C_\varepsilon(t_1)\cap \partial C_\varepsilon(t_2)= \varnothing$,we are to show that $N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}>N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$, for $t_2>t_1$ and for all $\phi=(\phi_1,\phi_2)\in Y^*\times Z^*$. Let $\phi=(\phi_1,\phi_2)\in Y^*\times Z^*$be a functional over $Y\times Z$. Hence corresponding to $\phi=(\phi_1,\phi_2)$, we can find by Proposition 4, a point $(\xi_1,\eta_1)\in\partial C_\varepsilon(t_1)$ and a point $(\xi_2,\eta_2)\in\partial C_\varepsilon(t_2)$ such that $\phi=(\phi_1,\phi_2)$ is the supporting hyper plane at $(\xi_1,\eta_1)\in\partial C_\varepsilon(t_1)$ and at $(\xi_2,\eta_2)\in\partial C_\varepsilon(t_2)$.Then $\langle(\xi_1,\eta_1),(\phi_1,\phi_2)\rangle= N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$ and $\langle(\xi_2,\eta_2),(\phi_1,\phi_2)\rangle$ $N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}$. Now, $C_\varepsilon(t_1) \subseteq C_\varepsilon(t_2)$ and $\partial C_\varepsilon(t_1)\cap \partial C_\varepsilon(t_2)=\phi$; hence $(\xi_1,\eta_1)\in \text{Int}\, \partial C_\varepsilon(t_2)$. Thus $\langle(\xi_1,\eta_1),(\phi_1,\phi_2)\rangle < \langle(\xi_2,\eta_2),(\phi_1,\phi_2)\rangle$, or, $N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}> N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+ \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$, or, $N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}> N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}$

Proposition 9: Let $(\xi,\eta)\in C_\varepsilon(t_f)\cap\partial C_\varepsilon(t_f)$ for some $t=t_f$ and let $t \geq t_f$ .Then

$$\max_{\phi=(\phi_1,\phi_2)\neq(0,0)\in(Y\times Z)'} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_t\phi_1 +S'_t\phi_2,f_t) : f_t\in X'_t\} + \varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3\in Y'\}}$$

is a non-increasing function for $t \geq t_f$.

Proof: Let $t_f< t_1< t_2$. Then, from theorem 8,

$$\max_{\phi=(\phi_1,\phi_2)} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}}$$
$$\frac{\langle(\xi,\eta),(\phi_1',\phi_2')\rangle}{N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}} \quad \rightarrow(i)$$

where $\phi'=(\phi_1',\phi_2')$ define the supporting hyper plane to point of intersection of the ray through (ξ,η) with $\partial C_\varepsilon(t_1)$. Denote this point by (ξ_1,η_1). Then $(\xi_1,\eta_1)= l_1(\xi,\eta)$ for some $l_1\geq1$.Let $u_{t1}\in U_{Xt1}$ be the optimal control to reach (ξ_1,η_1) i.e. $(\xi_1,\eta_1)= \overline{T}_{t1}(u_{t1})$. Since $\overline{T}_{t1}$ is the restriction of $\overline{T}_{t2}$ on U_{Xt1} we have $(\xi_1,\eta_1)= \overline{T}_{t1}(u_{t1})= \overline{T}_{t2}(u_{t2})$ -----›(ii). By Proposition 4 we also have $\langle(\xi_1,\eta_1),(\phi_1',\phi_2')\rangle= N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}$. Then from (i) we get

$$\max_{\phi=(\phi_1,\phi_2)} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}} =$$

$$\frac{\langle(\xi,\eta),(\phi_1',\phi_2')\rangle}{N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}}=\frac{\langle(\xi,\eta),(\phi_1',\phi_2')\rangle}{\langle(\xi_1,\eta_1),(\phi_1',\phi_2')\rangle}=$$

$$\frac{\langle(\xi,\eta),(\phi_1',\phi_2')\rangle}{\langle \bar{T}_{t1}(u_{t1}),(\phi_1',\phi_2')\rangle}=\frac{\langle(\xi,\eta),(\phi_1',\phi_2')\rangle}{\langle \bar{T}_{t2}(u_{t1}),(\phi_1',\phi_2')\rangle} \quad \text{-------->(iii)}$$

Again, let

$$\max_{\phi=(\phi_1,\phi_2)}\frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}+N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}}=$$

$$\frac{\langle(\xi,\eta),(\phi''_1,\phi''_2)\rangle}{N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}+N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}}$$

where $(\phi''_1,\phi''_2)\in Y^*\times Z^*$ define the supporting hyperplane to $(\xi_2,\eta_2)=l_2(\xi_1,\eta_1)$ for some $l_2\geq l_1$ and $(\xi_2,\eta_2)\in\partial C_\varepsilon(t_2)$. Similarly, as before,

$$\max_{\phi=(\phi_1,\phi_2)}\frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}+N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}}=$$

$$\frac{\langle(\xi,\eta),(\phi''_1,\phi''_2)\rangle}{N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}+N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}}=\frac{\langle(\xi,\eta),(\phi''_1,\phi''_2)\rangle}{\langle T_{t2}(u_{t2}),(\phi''_1,\phi''_2)\rangle} \quad \text{----(iv)}$$

Where $u_{t2}\in U_{Xt2}\subset X_{t2}$ is optimal control to reach at (ξ_2,η_2)

Now, $(\xi_1,\eta_1)\in C_\varepsilon(t_1)=\bar{T}_{t1}(U_{Xt1})=\bar{T}_{t2}(U_{Xt1})\subset\bar{T}_{t2}(U_{Xt2})=C_\varepsilon(t_1)$.

Since (ϕ''_1,ϕ''_2) is the supporting hyperplane to at $C_\varepsilon(t_2)$ at (ξ_2,η_2), we have $\langle(\xi_1,\eta_1),(\phi''_1,\phi''_2)\rangle\leq\langle(\xi_2,\eta_2),(\phi''_1,\phi''_2)\rangle=\langle\bar{T}_{t2}(u_{t2}),(\phi''_1,\phi''_2)\rangle$.

Thus $\langle\bar{T}_{t2}(u_{t1}),(\phi''_1,\phi''_2)\rangle\leq\langle\bar{T}_{t2}(u_{t2}),(\phi''_1,\phi''_2)\rangle$ ----->(v) [by (ii)].
But $\langle T_{t2}(u_{t1}),(\phi''_1,\phi''_2)\rangle=\langle(\xi_1,\eta_1),(\phi''_1,\phi''_2)=1/l_1l_2\langle(\xi_2,\eta_2),(\phi''_1,\phi''_2)\rangle>0$ ---->(vi)
Since $(0,0)\in\text{Int }C_{\alpha2}(t_2)$ and $1/l_2>0$. This also follows from the fact that $\langle(\xi_2,\eta_2),(\phi''_1,\phi''_2)\rangle=N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}+N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}$ by Proposition 4.
Also, $\langle(\xi,\eta),(\phi''_1,\phi''_2)\rangle=(1/l_1l_2)\langle(\xi_2,\eta_2),(\phi''_1,\phi''_2)\rangle>0$, ---->(vii)

Hence from (v),(vi) and (vii)

$$\frac{\langle(\xi,\eta),(\phi''_1,\phi''_2)\rangle}{\langle T_{t2}(u_{t1}),(\phi''_1,\phi''_2)\rangle}\geq\frac{\langle(\xi,\eta),(\phi''_1,\phi''_2)\rangle}{\langle T_{t2}(u_{t2}),(\phi''_1,\phi''_2)\rangle} \quad \text{---->(viii)}$$

Now from (iii) and (viii)

$$\max_{\phi=(\phi_1,\phi_2)} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_{t2}\phi_1+S'_{t2}\phi_2,f_{t2}):f_{t2}\in X'_t\}+N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}} =$$

$$\frac{\langle(\xi,\eta),(\phi''_1,\phi''_2)\rangle}{\langle T_{t2}(u_{t2}),(\phi_1'',\phi_2'')\rangle} \geq \frac{\langle(\xi,\eta),(\phi''_1,\phi'')\rangle}{\langle T_{t2}(u_{t1}),(\phi''_1,\phi''_2)\rangle} \qquad \text{-----(ix)}$$

Since maximum is attained at (ϕ''_1,ϕ''_2). Now using (viii) and (ix) we have

$$\max_{\phi=(\phi_1,\phi_2)} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}} \geq \frac{\langle(\xi,\eta),(\phi''_1,\phi''_2)\rangle}{\langle T_{t2}(u_{t1}),(\phi''_1,\phi''_2)\rangle} \geq$$

$$\frac{\langle(\xi,\eta),(\phi''_1,\phi''_2)\rangle}{\langle \bar{T}_{t2}(u_{t2}),(\phi''_1,\phi''_2)\rangle} = \max_{\phi=(\phi_1,\phi_2)} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_{t1}\phi_1+S'_{t1}\phi_2,f_{t1}):f_{t1}\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}}$$

Corollary 3:

$$\max_{\phi=(\phi_1,\phi_2)} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T'_t\phi_1+S'_t\phi_2,f_t):f_t\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}}$$

is a non-increasing function of t for $t \geq 0$.

We now turn our attention to the investigation to the Problem II. In the present setting, we shall consider the set $C_\varepsilon(\rho,t)=\{\rho\widehat{T}_t(U_{Xt})+(\varepsilon U_Y\times U_Z)\}$, $(\varepsilon>0)$, $t\in(0,\infty)$.Here both S_t and T_t are into mapping.

The corresponding auxiliary problem is as follows:

Let $(\xi,\eta)\in\partial C_\varepsilon(\rho,t)$ with $N_2\{(\xi-T_tu_t,w):w\in Y\}\leq\varepsilon$ for some given time t. Then determine $u_t\in U_{Xt}\subset X_t$,such that $\widehat{T}_t(u_t)=(\xi,\eta)$ and $N_4\{(\eta-S_tu_t,w_1):w_1\in Z\}$ is minimum. Such an u_t (if it exists) will be called an optimal control and t is the corresponding minimum time to reach (ξ,η).

Most of the arguments we develop are parallel to those in previous section.

Definition 8 :$\xi\in Y$ is called (ε,ρ)-regular(with respect to T_t) if their exits at least one element $u_t \in \rho U_{Xt}$ such that $N_2\{(\xi-T_tu_t,w):w\in Y\}\leq\varepsilon$.

Proposition 10: Let ξ be an (ε,ρ)-regular element and suppose that $(\xi,\eta)\in\partial C_\varepsilon(\rho,\alpha)$. Then any hyperplane $(\phi_1,\phi_2)(\neq(0,0))$ of support of $C_\varepsilon(\rho,t)$ at (ξ,η) satisfies the following relations:
(i)$\langle(\xi,\eta),(\phi_1,\phi_2)\rangle\leq\rho N'_1\{(T'_t\phi_1+S'_t\phi_2,f_t):f_t\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}+N'_4\{(\phi_2,\phi_4):\phi_2,\phi_4\in Z'\}$, (ii)$N'_4\{(\phi_2,\phi_4):\phi_2,\phi_4\in Z'\}\neq0$.

Proof: (i) can be shown as in previous section. We shall proved (ii). If possible, let $\phi_2=0$. Then $\phi_1\neq0$ and for all $u_t\in X_t$, $N_1\{(u_t,v_t):u_t,v_t\in X_t\}$. $N'_1\{(T'_t\phi_1,f_t):f_t\in X'_t\}+N_2\{(\xi-T_tu_t,w):w\in Y\}$. $N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}\geq\langle\xi,\phi_1\rangle\geq\rho N'_1\{(T'_t\phi_1,f_t):f_t\in X'_t\}+\varepsilon N'_2\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}$. Hence

$[N_2\{(\xi - T_t u_t, w):w\in Y\} - \varepsilon] .N'_2\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\}\geq[\rho - N_1\{(u_t,v_t):u_t,v_t\in X_t\}]$. $N'_1\{(T'_t\phi_1,f_t):f_t\in X'_t\}$ for all $u_t \in\rho U_{Xt}$, which is contrary to the assumption. Hence the result.

Corollary 4: ξ is an(ε, ρ)-regular element if and only if $\langle\xi,\phi\rangle < \rho\, N'_1\{(T_t'\phi,f_t):f_t \in X'_t\} + \varepsilon\, N'_2\{(\phi, \phi_3): \phi,\phi_3 \in Y'\}$ for all $\phi(\neq 0) \in Y'$. In the following results, it is always assumed that ξ is an (ε, ρ)-regular element w.r.to T_t.

Proposition 11: Suppose that $(\xi, \eta) \in \partial C_\varepsilon(\rho, \alpha)$. Then for all $u \in \rho U_x$ satisfying $N_2 \{(\xi - T_t u_t,w):w\in Y\} \leq \varepsilon$, we have $N_4\{(\eta - S_t u_0, w_1):w_1\in Z\} \geq 1$.

Proof : With (ϕ_1, ϕ_2) defined in the previous property, we have, for all $u_t \in\rho U_{xt}\subset X_t$, $\langle(\xi, \eta), (\phi_1, \phi_2)\rangle=\langle(\xi - T_t u_t + T_t u_t, \eta - S_t u_t + S_t u_t), (\phi_1, \phi_2)\rangle=\langle\xi - T_t u_t, \phi_1\rangle + \langle\eta - S_t u_t,\phi_2\rangle + \langle u_t, T_t'\phi_1+S_t'\phi_2\rangle$
$\leq N_1\{(u_t, v_t): u_t, v_t\in X_t\}\, N_1'\{(T_t' \phi_1 + S_t'\phi_2, f_t): f_t\in X_t'\} + N_2\{(\xi - T_t u_t,w): w\in Y \}\, N_2'\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\} + N_4\{(\eta - S_t u_t, w_1): w_1 \in Z\}\, N_4'\{(\phi_2, \phi_4): \phi_2, \phi_4 \in Z'\}$.
Now, $N_1\{(u_t, v_t): u_t, v_t\in X_t\}\, N_1'\{(T_t' \phi_1 + S_t'\phi_2, f_t): f_t\in X_t'\} + N_2\{(\xi - T_t u_t,w): w\in Y \}\, N_2'\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\} + N_4\{(\eta - S_t u_t, w_1): w_1 \in Z\}\, N_4'\{(\phi_2, \phi_4): \phi_2, \phi_4\in Z'\} \geq \langle(\xi, \eta), (\phi_1, \phi_2)\rangle$
$\geq\rho\, N_1'\{(T_t' \phi_1 + S_t'\phi_2, f_t): f_t\in X_t'\} + \varepsilon\, N_2'\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\} + N_4'\{(\phi_2, \phi_4): \phi_2, \phi_4 \in Z'\}$.
Hence $(N_4\{(\eta - S_t u_t, w_1): w_1 \in Z \} - 1)\, N_4'\{(\phi_2, \phi_4): \phi_2, \phi_4 \in Z'\} \geq(\rho - N_1\{(u_t, v_t): u_t, v_t\in X_t\})$
$N_1'\{(T_t' \phi_1 + S_t'\phi_2, f_t): f_t\in X_t'\} + (\varepsilon - N_2\{(\xi - T_t u_t, w): w\in Y \}\, N_2'\{(\phi_1, \phi_3): \phi_1, \phi_3 \in Y'\} \geq 0$,
$N_4\{(\eta - S_t u_t, w_1): w_1 \in Z \} \geq 1$. Hence the result.
Similar properties as in the previous section can be proved for this problem also. To find out the optimal solution we further need the following definition.

Definition 9: We shall say that $\eta \in Z$ is normal (with respect to $(\widehat{T}_t ,\xi, \varepsilon, \rho)$) if either

$$\inf_{N_1\{(u_t, v_t): u_t, v_t\in X_t\} \leq \rho} \{N_4\{(\eta - S_t u_t, w_1): w_1 \in Z \}\}> \inf_{u_t \in X_t} \{N_4\{(\eta - S_t u_t, w_1): w_1 \in Z \}\} \text{ holds .}$$

or,
$$\inf_{N_2\{(\xi - T_t u_t,w): w\in Y \} \leq \varepsilon} \{N_4\{(\eta - S_t u_t, w_1): w_1 \in Z \}\}> \inf_{u_t \in X_t} \{N_4\{(\eta - S_t u_t, w_1): w_1 \in Z \}\} \text{ holds.}$$

Definition 10: We shall say that $\eta \in Z$ is (ε, ρ) – normal (with respect to $(\widehat{T}_t, \xi)$) if

$$\inf_{\substack{N_1\{(u_t, v_t): v_t\in X_t\} \leq \rho \\ N_2\{(\xi - T_t u_t,w): w\in Y \} \leq \varepsilon}} \{N_4\{(\eta - S_t u_t, w_1): w_1 \in Z \}\}> \inf_{N_2\{(\xi - T_t u_t,w): w\in Y \} \leq \varepsilon} \{N_4\{(\eta - S_t u_t, w_1): w_1 \in Z \}\} \text{ holds.}$$

Note 2: One can easily verified that $N_1'\{(T_t'\phi_1 + S_t'\phi_2, f_t): f_t\in X_t'\} \neq 0$, for each (ε, ρ)- normal element $\eta\in Z$.

Proposition 12: Assume that either (i) or (ii) stated in the corollary 1 holds. Then there exists a solution to problem II for each (ε, ρ) regular element ξ. Suppose, further, that η is a normal element. Then u_0 is an optimal solution if and only if u_0 takes the form
$u_0 = \rho\overline{(T_t'\phi_1 + S_t'\phi_2)}$,Where (ϕ_1, ϕ_2) of norm 1 is determined by either of the following:

$\xi = \rho T_t(\overline{T_t'\phi_1 + S_t'\phi_2}) + \varepsilon \overline{\phi}_1$, $\eta = \rho S_t(\overline{T_t'\phi_1 + S_t'\phi_2}) + \{(\langle\xi, \phi_1\rangle + \langle\eta, \phi_2\rangle - \rho N_1'\{(T_t'\phi_1 + S_t'\phi_2, f_t)$: $f_t \in X_t'\} - \varepsilon N_2'\{(\phi_1, \phi_3):\phi_1,\phi_3 \in Y'\})/N_4'\{(\phi_2, \phi_4): \phi_2, \phi_4 \in Z'\}\} \overline{\phi}_2$,

$$\max_{N3'\{(\phi1,\phi2),\ \phi2\neq 0\}=1} \frac{\{\langle\xi,\phi_1\rangle+\langle\eta,\phi_2\rangle-\rho N_1'\{(T_t'\phi_1+S_t'\phi_2,f_t):f_t\in X_t'\}-\varepsilon N_2'\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}}{N_4'\{(\phi_2, \phi_4): \phi_2, \phi_4 \in Z'\}} = 1$$

Example 1: Let X be real inner product space. Then X is asymmetric generalized 2-normed space under the 2-norm $N_1(x,y)=\sum_{i=1}^{n} |\langle x_i,y_i\rangle| = \sum_{i=1}^{n} x_i y_i$, $\forall\ x_i,y_i \in X$, by definition 7.

Example 2: In order to show an application of the theory, let us consider a minimum effort control problem. Consider the dynamical system governed by the linear differential equation:

$$\frac{dx}{dt}(t)=Ax(t)+Bu(t),\ x\in X, \text{ X is a normed linear space.}$$

Where x(t) is n×1 state vector, u(t) is an r×1 control vector and A,B are constant matrices of appropriate dimensions. A control vector u(t) which satisfies $|u_j(t)|\le\rho$, j=1,2,…,r will be called admissible.

The problem is to find an admissible control vector u(t), such that the trajectories described by the system under U(t) remain within ε-neighbourhood of the target state x^d, i.e. $N_1\{(x(t_1)-x^d,u): x(t_1)-x^d,u\in X\} \le\varepsilon$, while minimizing the functional $J(u)=\int_{t0}^{t1}\sum_{j=1}^{r}|u_j(t)|.\int_{t0}^{t1}\sum_{j=1}^{r}|u_j(t)|$ where t_0,t_1 being initial and final times respectively. We assumed that there exists at least one admissible control u(t) which satisfies $N_1\{(x(t_1)-x^d,u): x(t_1)-x^d,u\in X\} <\varepsilon$ (i.e., (ε, ρ) regularity assumption).

Let us now specify the basic function spaces and linear operators as follows: $X= B_{\infty,\infty}^{(r)} \times B_{\infty,\infty}^{(r)} = L_\infty(l_\infty(r),\tau)\times L_\infty(l_\infty(r),\tau)$, $Y=l_\infty(\eta)\times l_\infty(\eta)$, $Z= B_{1,1}^{(r)}\times B_{1,1}^{(r)}= L_1(l_1(r),\tau)\times L_1(l_1(r),\tau)$, when $\tau=[t_0,t_1]$. Then by definition (7) X, Y, Z are generalized 2-normed spaces. T:X→Y defined by $Tu=\int_{t0}^{t1} e^{A(t1-s)}Bu(s)ds$ and S:X→Z defined by Su= − u, natural embedding of X into Z. Taking $\xi=X^d - e^{-A(t1-t0)}X(t_0)$, $\eta=Tu_0= - u_0$. The auxiliary problem becomes finding u, such that $N_2\{(\xi -Tu(.),w): \xi -Tu(.), w\in Y\}\le\varepsilon$, J(u) is minimized.

By applying property, we obtain $u_0=\overline{(S'\phi_1-\phi_2)}$, $\xi=T(\overline{T'\phi_1-\phi_2})+\varepsilon\overline{\phi}_1$, $\eta=Su_0= - u_0= - (\overline{T'\phi_1-\phi_2})$

$$\max_{(\phi_1,\phi_2)\neq(0,0)\in(Y\times Z)'} \frac{\langle(\xi,\eta),(\phi_1,\phi_2)\rangle}{N_1'\{(T(T'\phi_1-\phi_2), f_t): f_t\in X_t'\}+\varepsilon N_2'\{(\phi_1,\phi_3):\phi_1,\phi_3\in Y'\}} = 1.$$

For another examples see U.Adak([9],[10]).

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