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Locally Conformal Kahler Manifold of Pointwise Holomorphic Sectional Curvature Tensor

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Abstract

In the present paper we found the necessary condition in which a locally conformal Kahler manifold is a manifold of a pointwise holomorphic sectional curvature tensor. It has been proved that, if M is a Locally conformal Kahler manifold of the pointwise holomorphic sectional curvature tensor and projective(conformal) flat with J-invariant Ricci tensor, then M is an Einstein manifold.

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1. Introduction

Fifty years ago, a number of researchers studied one of the most important subjects of differential geometry, whose application is used in the synthesis of the differential geometrical structure, it is called "Almost Hermitian manifold". This subject is classified into different components for an attempt to determine its specifications and features accurately.

The first practical study was done by Koto 1960 [10], upon the findings of which had been depended by Gray to set forth a number of examples on practical manifolds in 1965 [3]. A new study about the kinds of almost Hermitian manifold was conducted by Gray and Hervella in 1980 [4], they found that the effect of the unitary group $U(n)$ on the space of tensors of type (3,0) was reducible. Accordingly, the space decomposed into four irreducible component spaces. Thus, this effect defined sixteen subspaces of the space of type (3,0). This study was done by using Kozal's operator method [11]. A great contribution was made by the Russian researcher Kirichenko [5], who studied the almost Hermitian manifold by the adjoint *G*-structure space. Kirichenko found two tensors, which helped the researchers to study the different properties of almost Hermitian manifold. In 2001, Banaru [2] succeeded to classify the sixteen classes of almost Hermitian manifold by using the two tensors of Kirichenko, which are called the Kirichenko's tensors. Abood [1] studied the properties of these tensors.

The locally conformal Kahler manifold which are going to be dealt with in this study, is one of the sixteen classes of almost Hermitian manifold. The first study on locally conformal Kahler manifold was conducted by Libermann 1955 [12]. Vaisman, in 1981 put down some geometrical conditions for locally conformal Kahler manifold [19]. Letter on in1982, Tricerri mentioned different examples about the locally conformal Kahler manifold[18].

In each Riemannian manifold, there is a tensor of the type (3,1), which is invariant with respect to the metric transformation. This tensor is called as conformal curvature tensor or Wely's tensor [15], which is going to be considered in this study.

2. Preliminaries

Definition 2.1 [9]. A tensor field J of type (1,1) on a smooth manifold M is called an almost complex structure (AC-structure), if at each point $p \in M$, there is an endomorphism J_p of a tangent space $T_p(M)$, such that $J_p^2 = -id$, where id is the identity mapping of $T_p(M)$.

A smooth manifold with an almost complex structure is called an almost complex manifold (AC -manifold).

Definition 2.2 [9]. Let M be a smooth manifold and $X(M)$ be a module of vector fields on M. An almost Hermitian structure (AH-structure) on M is a pair $\{J, g = \langle , \rangle\}$, where J is an AC-structure, g is the (pseudo) Riemannian metric on M such that:

$\langle JX, JY \rangle = \langle X, Y \rangle$, $\forall\ X, Y \in X(M)$.

A smooth manifold with an AH -structure is called an almost Hermitian manifold (AH - manifold) and will be denoted by $\{M, J, g = \langle , \rangle\}$.

Lemma 2.3 [9]. Every almost complex manifold has an even dimension and is orientable.

Definition 2.4 [8]. Let M be an almost complex manifold of real dimension $2n$, the module $X^c(M)=\mathbb{C}\otimes X(M)=\left\{X : X=\sum_{k=1}^{N} Z_k\otimes X_k, Z_k\in\mathbb{C}, X_k\in X(M)\right\}$ is called a complexification of the module $X(M)$.

Definition 2.5 [13]. An endomorphism $\tau : X^c(X)\to X^c(X)$ which is defined by: $\tau(X)=\tau(\sum_k Z_k\otimes X_k)=\sum_k \overline{Z}_k\otimes X_k\in X^c(M)$ for all $X=\sum_k Z_k\otimes X_k$ is called an operator of complex conjugate.

Remark [7]. In the module $X^c(M)$ we can define two complementary projections: $\sigma : X^c(M)\to X^c(M)$ by the form $\sigma=\frac{1}{2}(id-\sqrt{-1}J^c)$ and $\overline{\sigma} : X^c(M)\to X^c(M)$ by the form $\overline{\sigma}=\frac{1}{2}(id+\sqrt{-1}J^c)$, where $J^c=id\otimes J$ is the linear complex extension of endomorphism J.
Let us denote $\operatorname{Im}\sigma=D$ and $\operatorname{Im}\overline{\sigma}=\overline{D}$, then $X^c(M)=D+\overline{D}$, where D and $\overline{D}$ are respectively proper submodules of the endomorphism J^c with proper values $\sqrt{-1}$ and $-\sqrt{-1}$, then each $X\in X^c(M)$ can be written by the form $X=\sigma(X)+\overline{\sigma}(X)$.

Theorem 2.6 [5]. If M is an AH- manifold, then the basis of $T_p(M)$ given by the form $\{e_1,...,e_n,Je_1,...,Je_n\}$.

Definition 2.7 [5]. The basis $\{e_1,...,e_n,Je_1,...,Je_n\}$ is called a real adept basis of AH-structure $\{J,g\}$, or RA-basis .

By using this basis we can construct a new basis $\{\varepsilon_1,...,\varepsilon_n,\overline{\varepsilon}_1,...,\overline{\varepsilon}_n\}$, where $\varepsilon_a=\sigma(e_a)$ and $\overline{\varepsilon}_a=\overline{\sigma}(e_a)$. This basis is called a basis of AH-structure or A-basis. The corresponding frame is $\{p,\varepsilon_1,...,\varepsilon_n,\overline{\varepsilon}_1,...,\overline{\varepsilon}_n\}$ which is called A-frame [5].

We will use the notations $\{\varepsilon_{\hat{1}}=\overline{\varepsilon}_1,...,\varepsilon_{\hat{n}}=\overline{\varepsilon}_n\}$. Suppose that the indices $i,j,k,\ell,...$ in the range $1,...,2n$ and the indices $a,b,c,d,e,f,g,h,...$ in the range $1,...,n$ Denote $\hat{a}=a+n$, then A-frame can be written as the form $\{p,\ \varepsilon_1,\dots,\varepsilon_n,\varepsilon_{\hat{1}},\dots,\varepsilon_{\hat{n}}\}$.

Remark [5]. It is well known, that the given AH-structure on manifold M equivalent to the given G-structure in the principle fiber bundle of all complex frames of manifold M with structure group $U(n)$, this group is called an adjoint G-structure.

The space consists of A-frames characterizing , that the components matrices of J and g in this A-frames have the following forms:

$$(J^i_j)=\begin{pmatrix}\sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n\end{pmatrix} \quad , \quad (g_{ij})=\begin{pmatrix}0 & I_n \\ I_n & 0\end{pmatrix} \tag{2.1}$$

where I_n is the unit matrix of order n.

3. Locally conformal Kahler manifold

Definition 3.1 [15]. Let g and $\tilde{g}$ be two Riemannian metrics on smooth manifold M, we say that on M given a conformal transformation metric if there is a smooth function $f \in C^{\infty}(M)$ such that $\tilde{g} = e^{2f} g$.

Let $\{M, J, g =<,>\}$ be an *AH*-manifold , if there exists a conformal transformation of the metric g into the metric $\tilde{g}$, then $\{M, J, \tilde{g} = e^{2f} g\}$ will be *AH*-manifold. In this case we say that on smooth manifold M given conformal transformation of *AH*-structure, denoted by $\tilde{M}_f$.

Definition 3.2 [4]. An *AH*-manifold is called a locally conformal Kahler manifold, if for each point $m \in M$ there exists an open neighborhood U of this point and there exists $f \in C^{\infty}(U)$, such that $\tilde{U}_f$ is Kahler manifold. We will denote to the locally conformal Kahler manifold by *L.C.K*-manifold.

Definition 3.3 [4]. Let M be an *AH*-manifold, the form which is given by the relation $\alpha = \frac{-1}{n-1} S\ \Omega \circ J$ is called a Lie form , where S represents the coderivative. If Ω is $r-$form, then its coderivative $S\ \Omega$ is $(r-1)-$form.

Remark [2]. By the Banaru's classification of *AH*-manifold, the *L.C.K.*- manifold satisfies the following condition:

$B^{abc} = 0 \quad , \quad B^{ab}_c = \alpha^{[a}\delta^{b]}_c$.

Theorem 3.4 [16]. In the adjoint G – structure space, the components of Riemannian curvature tensor of *L.C.K*-manifold are given by the following forms:

1. $R^a_{bcd} = \alpha_{a[c}\delta^b_{d]} + \frac{1}{2}\alpha_a \alpha_{[c}\delta^b_{d]}$

2. $R^{\hat{a}}_{\hat{b}\hat{c}\hat{d}} = -\alpha^{a[c}\delta^{d]}_{b} - \frac{1}{2}\alpha^{a}\alpha^{[c}\delta^{d]}_{b}$
3. $R^{a}_{\hat{b}cd} = -2\alpha^{[a}_{[c}\delta^{b]}_{d]}$
4. $R^{\hat{a}}_{b\hat{c}\hat{d}} = 2\alpha^{[c}_{[a}\delta^{d]}_{b]}$
5. $R^{a}_{bc\hat{d}} = A^{ad}_{bc} - \alpha^{[a}\delta^{h]}_{c}\alpha_{[h}\delta^{d}_{b]}$
6. $R^{\hat{a}}_{\hat{b}\hat{c}d} = -A^{bc}_{ad} + \alpha^{[h}\delta^{b]}_{d}\alpha_{[a}\delta^{c}_{h]}$
7. $R^{a}_{b\hat{c}d} = A^{ad}_{bc} - \alpha^{[a}\delta^{h]}_{d}\alpha_{[b}\delta^{c}_{h]}$
8. $R^{\hat{a}}_{\hat{b}c\hat{d}} = -A^{bc}_{ad} + \alpha^{[b}\delta^{h]}_{c}\alpha_{[a}\delta^{d}_{h]}$
9. $R^{a}_{b\hat{c}\hat{d}} = \alpha^{a[c}\delta^{d]}_{b} + \frac{1}{2}\alpha^{a}\alpha^{[c}\delta^{d]}_{b}$
10. $R^{\hat{a}}_{\hat{b}cd} = -\alpha_{a[c}\delta^{b}_{d]} - \frac{1}{2}\alpha_{a}\alpha_{[c}\delta^{b}_{d]}$
11. $R^{a}_{\hat{b}\hat{c}d} = -\alpha^{[a|c|}\delta^{b]}_{d} + \alpha^{[a}\delta^{b]}_{h}\alpha^{[h}\delta^{c]}_{d}$
12. $R^{\hat{a}}_{bc\hat{d}} = \alpha_{[a|c|}\delta^{d}_{b]} - \alpha_{[a}\delta^{h}_{b]}\alpha_{[h}\delta^{d}_{c]}$
13. $R^{a}_{\hat{b}c\hat{d}} = -\alpha^{[a|d|}\delta^{b]}_{c} + \alpha^{[a}\delta^{b]}_{h}\alpha^{[h}\delta^{d]}_{c}$
14. $R^{\hat{a}}_{b\hat{c}d} = \alpha_{[a|d|}\delta^{c}_{b]} - \alpha_{[a}\delta^{h}_{b]}\alpha_{[h}\delta^{c}_{d]}$
15. $R^{a}_{\hat{b}\hat{c}\hat{d}} = 0$
16. $R^{\hat{a}}_{bcd} = 0$

Where A^{ad}_{bc} are the components of holomorphic sectional curvature tensor[6].

Definition 3.5 [15]. A Ricci tensor is a tensor of type (2,0) which is defined by:
$r_{ij} = R^{k}_{ijk} = g^{kl}R_{kijl}$.

Theorem 3.6. In the adjoint G – structure space , the components of the Ricci tensor of *L.C.K*-manifold are given by the following forms:
1. $r_{ab} = \alpha_{c[b}\delta^{a}_{c]} + \frac{1}{2}\alpha_{c}\alpha_{[b}\delta^{a}_{c]} + \alpha_{[c|b|}\delta^{c}_{a]} - \alpha_{[c}\delta^{h}_{a]}\alpha_{[h}\delta^{c}_{b]}$
2. $r_{\hat{a}\hat{b}} = -\alpha^{c[b}\delta^{c]}_{a} - \frac{1}{2}\alpha^{c}\alpha^{[b}\delta^{c]}_{a} - \alpha^{[c|b|}\delta^{a]}_{c} + \alpha^{[c}\delta^{a]}_{h}\alpha^{[h}\delta^{b]}_{c}$
3. $r_{a\hat{b}} = 2\alpha^{[b}_{[c}\delta^{c]}_{a]} + A^{cc}_{ab} - \alpha^{[c}\delta^{h]}_{c}\alpha_{[a}\delta^{b}_{h]}$
4. $r_{\hat{a}b} = -2\alpha^{[c}_{[b}\delta^{a]}_{c]} - A^{ab}_{cc} + \alpha^{[a}\delta^{h]}_{b}\alpha_{[c}\delta^{c}_{h]}$

Proof. By using the definition 3.5 and theorem3.4, we have:
1. $r_{ab} = R^{k}_{abk}$
$\quad = R^{c}_{abc} + R^{\hat{c}}_{ab\hat{c}}$

$$r_{ab} = \alpha_{c[b}\delta^{a}_{c]} + \tfrac{1}{2}\alpha_c\alpha_{[b}\delta^{a}_{c]} + \alpha_{[c|b|}\delta^{c}_{a]} - \alpha_{[c}\delta^{h}_{a]}\alpha_{[h}\delta^{c}_{b]}$$

2. $r_{\hat{a}\hat{b}} = R^{k}_{\hat{a}\hat{b}k}$

$= R^{\hat{c}}_{\hat{a}\hat{b}\hat{c}} + R^{c}_{\hat{a}\hat{b}c}$

$r_{\hat{a}\hat{b}} = -\alpha^{c[b}\delta^{c]}_{a} - \tfrac{1}{2}\alpha^{c}\alpha^{[b}\delta^{c]}_{a} - \alpha^{[c|b|}\delta^{a]}_{c} + \alpha^{[c}\delta^{a]}_{h}\alpha^{[h}\delta^{b]}_{c}$

3. $r_{a\hat{b}} = R^{k}_{a\hat{b}k}$

$= R^{\hat{c}}_{a\hat{b}\hat{c}} + R^{c}_{a\hat{b}c}$

$r_{a\hat{b}} = 2\alpha^{[b}_{[c}\delta^{c]}_{a]} + A^{cc}_{ab} - \alpha^{[c}\delta^{h]}_{c}\alpha_{[a}\delta^{b}_{h]}$

4. $r_{\hat{a}b} = R^{k}_{\hat{a}bk}$

$= R^{c}_{a\hat{b}c} + R^{\hat{c}}_{ab\hat{c}}$

$r_{\hat{a}b} = -2\alpha^{[c}_{[b}\delta^{a]}_{c]} - A^{ab}_{cc} + \alpha^{[a}\delta^{h]}_{b}\alpha_{[c}\delta^{c}_{h]}$

Definition 3.7 [6]. A holomorphic sectional curvature (*HS*-curvature) tensor of an *AH*-manifold in the direction $X \in X(M),\ X \neq 0$ is a function $H(X)$ which is defined by $\langle R(X,JX)X,JX\rangle = H(X)\ \|X\|^4$.

Definition 3.8 [6]. A manifold M is called a manifold of a pointwise holomorphic sectional curvature (*PHS*-curvature) tensor, if H does not depend on X , that means,

$$\langle R(X,JX)X,JX\rangle = c\ \|X\|^4 \quad ; \quad X \in X(M)\ ,\ \ c \in C^{\infty}(M).$$

Lemma 3.9 [17]. If M is an *AH*-manifold of *PHS*-curvature tensor , then we have: $\|X\|^4 = 2\tilde{\delta}^{bc}_{ad}X^{a}X^{d}X_{b}X_{c}$, where $\tilde{\delta}^{bc}_{ad} = \tfrac{1}{2}(\delta^{b}_{a}\delta^{c}_{d} + \delta^{b}_{d}\delta^{c}_{a})$ is a Kroneker delta of the second type.

Lemma 3.10 [17]. If M is an *AH*-manifold of *PHS*-curvature tensor , then we have:

$$\begin{aligned}\langle R(X,JX)X,JX\rangle = (&R_{abcd}X^{a}X^{b}X^{c}X^{d} - 4R_{abc\hat{d}}X^{a}X^{b}X^{c}X^{\hat{d}} - 2R_{ab\hat{c}\hat{d}}X^{a}X^{b}X^{\hat{c}}X^{\hat{d}}\\ &- 4R_{a\hat{b}\hat{c}d}X^{a}X^{\hat{b}}X^{\hat{c}}X^{d} - 4R_{\hat{a}\hat{b}\hat{c}d}X^{\hat{a}}X^{\hat{b}}X^{\hat{c}}X^{d}\\ &+ R_{\hat{a}\hat{b}\hat{c}\hat{d}}X^{\hat{a}}X^{\hat{b}}X^{\hat{c}}X^{\hat{d}})\ .\end{aligned}$$

Theorem 3.11. If M is *L.C.K*-manifold of the *PHS* - curvature tensor , then the components of *HS*-curvature tensor in the a djoint *G*-structure space satisfies the condition $A^{bc}_{ad} = \alpha^{[h}\delta^{b]}_{d}\alpha_{[a}\delta^{c}_{h]} + \tfrac{c}{2}\tilde{\delta}^{bc}_{ad}$.

Proof. Let M be *L.C.K*-manifold of *PHS*-curvature tensor.

According to the definition 3.8 we have:

$$\left\langle R(X, JX)X, JX \right\rangle = c\ \|X\|^4; X \in X(M),\ \ c \in C^{\infty}(M) \tag{3.1}$$

According to the Lemmas 3.9, 3.10, and Theorem 3.4 , the equation (3.1) becomes:

$$\begin{aligned} &-4(\alpha_{[a|c|}\delta^{d}_{b]} - \alpha_{[a}\delta^{h}_{b]}\alpha_{[h}\delta^{d}_{c]})X^{a}X^{b}X^{c}X^{\hat{d}} - 4\alpha^{[c}_{[a}\delta^{d]}_{b]}X^{a}X^{b}X^{\hat{c}}X^{\hat{d}} \\ &+\alpha^{[h}\delta^{b]}_{d}\alpha_{[a}\delta^{c}_{h]})X^{a}X^{\hat{b}}X^{\hat{c}}X^{d} + 4(\alpha^{[a|c|}\delta^{b]}_{d} - \alpha^{[a}\delta^{b]}_{h}\alpha^{[h}\delta^{c]}_{d})X^{\hat{a}}X^{\hat{b}}X^{\hat{c}}X^{d} \\ &= 2c\tilde{\delta}^{bc}_{ad}X^{a}X^{d}X_{b}X \end{aligned} \tag{3.2}$$

Now, by symmetrization the equation (3.2) by (a,b), we get:

$$4(A^{bc}_{ad} - \alpha^{[h}\delta^{b]}_{d}\alpha_{[a}\delta^{c}_{h]})X^{a}X^{\hat{b}}X^{\hat{c}}X^{d} = 2c\tilde{\delta}^{bc}_{ad}X^{a}X^{d}X_{b}X_{c}\,.$$

Therefore $A^{bc}_{ad} = \alpha^{[h}\delta^{b]}_{d}\alpha_{[a}\delta^{c}_{h]} + \frac{c}{2}\tilde{\delta}^{bc}_{ad}$.

Definition 3.12 [17]. An *AH*-manifold has $J-$invariant Ricci tensor if $J \circ r = r \circ J$.

Lemma 3.13 [17]. An *AH*-manifold has $J-$invariant Ricci tensor if , and only if, in the a djoint *G*-structure space $r^{\hat{a}}_{b} = 0$.

Definition 3.14 [17]. A projective tensor of an *AH*-manifold is a tensor P of type (4,0) which is defined by the form:

$$P_{ijk\ell} = R_{ijk\ell} + \frac{1}{2n-1}(r_{ik}g_{j\ell} - r_{jk}g_{i\ell}) \tag{3.3}$$

Where r, R and g are respectively Ricci tensor, Riemannian curvature tensor and Riemannian metric.

This tensor has properties similar to the properties of Riemannian curvature tensor, that means $P_{ijk\ell} = -P_{jik\ell} = -P_{ij\ell k} = P_{k\ell ij}$.

Definition 3.15[14]. A Riemannian manifold is called an Einstein manifold, if the components of Ricci tensor satisfies the equation $r_{ij} = eg_{ij}$, where e and g are respectively an Einstein constant and Riemannian metric.

Definition 3.16. An *AH*-manifold is called a projective flat if the projective tensor is equal to zero.

Theorem 3.17. If M is *L.C.K*-manifold of the *PHS*-curvature tensor and is a projective flat with $J-$invariant Ricci tensor, then M is an Einstein manifold.

Proof. Suppose that M is *L.C.K*-manifold of the *PHS*-curvature tensor and projective flat with $J-$invariant Ricci tensor.

Put $i = a, j = \hat{b}, k = \hat{c}$ and $\ell = d,$ then the equation (3.3) becomes:

$$P_{a\hat{b}\hat{c}d} = R_{a\hat{b}\hat{c}d} + \frac{1}{2n-1}(r_{a\hat{c}}g_{\hat{b}d} - r_{\hat{b}\hat{c}}g_{ad})$$
$$= R_{a\hat{b}\hat{c}d} + \frac{1}{2n-1} r_{a\hat{c}}g_{\hat{b}d}$$

According to the definition 3.16 and theorems 3.4, 3.11 we obtained:

$$\tfrac{-c}{2}\tilde{\delta}^{bc}_{ad} + \tfrac{1}{2n-1}(r^c_a\delta^b_d) = 0$$

Thus

$$\tfrac{-c}{2}[\delta^b_a\delta^c_d + \delta^b_d\delta^c_a] + \tfrac{1}{2n-1}(r^c_a\delta^b_d) = 0 \qquad (3.4)$$

by contracting (3.4) by the indices (a,b) we obtained:

$r^c_d = e\delta^c_d$, where $e = \dfrac{c(2n-1)(n+1)}{2}$

Since Ricci tensor is $J-$invariant, so by Lemma 3.13 we get:

$r_{ij} = eg_{ij}$, where e is an Einstein constant.

Therefore M is an Einstein manifold.

Remark [15]. In each Riemannian manifold M (in particular *AH*-manifold) of dimension more than 2, we can define a tensor $W = \{W^i_{jk\ell}\}$ of type (3,1) which is invariant with respect to the conformal transformation metric .

This tensor is called as conformal curvature tensor of manifold M, or is called welye's tensor and defined by the form:

$$W_{ijk\ell} = R_{ijk\ell} + \tfrac{1}{2(n-1)}(r_{ik}g_{j\ell} + r_{j\ell}g_{ik} - r_{i\ell}g_{jk} - r_{jk}g_{i\ell}) + \frac{K(g_{jk}g_{i\ell} - g_{j\ell}g_{ik})}{(2n-1)(2n-2)} \qquad (3.5)$$

where R, r, g and K are respectively the Riemannian curvature tensor, Ricci tensor, Riemannian metric and scalar curvature tensor.

This tensor has properties similar to the properties of Riemannian curvature tensor, i.e. $W_{ijk\ell} = -W_{jik\ell} = -W_{ij\ell k} = W_{k\ell ij}$.

Lemma 3.18. In the adjoint $G-$ structure space, the components of Welye's tensor of *L.C.K*-manifold are given by the following forms:

1) $W_{abcd} = 0$

2) $W_{\hat{a}bcd} = \alpha_{a[c}\delta^b_{d]} + \tfrac{1}{2}\alpha_a\alpha_{[c}\delta^b_{d]} + \tfrac{1}{(n-1)}(r^{\hat{b}}_{[d}\delta^a_{c]})$

3) $W_{a\hat{b}cd} = -\alpha_{a[c}\delta^b_{d]} - \tfrac{1}{2}\alpha_a\alpha_{[c}\delta^b_{d]} + \tfrac{1}{(n-1)}(r^{\hat{a}}_{[c}\delta^b_{d]})$

4) $W_{ab\hat{c}d} = \alpha_{[a|d|}\delta^c_{b]} - \alpha_{[a}\delta^h_{b]}\alpha_{[h}\delta^c_{d]} + \tfrac{1}{(n-1)}(r^{\hat{d}}_{[b}\delta^c_{a]})$

5) $W_{abc\hat{d}} = \alpha_{[a|c|}\delta^d_{b]} - \alpha_{[a}\delta^h_{b]}\alpha_{[h}\delta^d_{c]} + \tfrac{1}{(n-1)}(r^{\hat{c}}_{[a}\delta^d_{b]})$

6) $W_{\hat{a}\hat{b}cd} = -2\alpha^{[a}_{[c}\delta^{b]}_{d]} + \frac{1}{(n-1)}(r^{a}_{[c}\delta^{b}_{d]} + r^{b}_{[d}\delta^{a}_{c]}) + \frac{K\delta^{ba}_{cd}}{(2n-1)(n-1)}$

7) $W_{\hat{a}b\hat{c}d} = A^{ad}_{bc} - \alpha^{[a}\delta^{h]}_{d}\alpha_{[b}\delta^{c}_{h]} - \frac{1}{(n-1)}r^{(a}_{(d}\delta^{c)}_{b)} + \frac{K\delta^{c}_{b}\delta^{a}_{d}}{(2n-1)(2n-2)}$

8) $W_{\hat{a}bc\hat{d}} = A^{ad}_{bc} - \alpha^{[a}\delta^{h]}_{c}\alpha_{[h}\delta^{d}_{b]} + \frac{1}{2(n-1)}(r^{a}_{c}\delta^{d}_{b} + r^{b}_{d}\delta^{a}_{c}) - \frac{K\delta^{b}_{d}\delta^{a}_{c}}{(2n-1)(2n-2)}$

And the others are conjugate to the above components.

Proof. By using,(2.1), theorem3.4 and the equation(3.5), we compute the components of conformal curvature tensor as the following:

1) Put $i = a$, $j = b$, $k = c$, and $\ell = d$, we obtained:

$$W_{abcd} = R_{abcd} + \frac{1}{2(n-1)}(r_{ac}g_{bd} + r_{bd}g_{ac} - r_{ad}g_{bc} - r_{bc}g_{ad}) + \frac{K(g_{bc}g_{ad} - g_{bd}g_{ac})}{(2n-1)(2n-2)}$$

$$= R_{abcd} = 0$$

2) Put $i = \hat{a}$, $j = b$, $k = c$, and $\ell = d$, we have:

$$W_{\hat{a}bcd} = R_{\hat{a}bcd} + \frac{1}{2(n-1)}(r_{\hat{a}c}g_{bd} + r_{bd}g_{\hat{a}c} - r_{\hat{a}d}g_{bc} - r_{bc}g_{\hat{a}d}) + \frac{K(g_{bc}g_{\hat{a}d} - g_{bd}g_{\hat{a}c})}{(2n-1)(2n-2)}$$

$$= R_{\hat{a}bcd} + \frac{1}{2(n-1)}(r^{\hat{b}}_{d}\delta^{a}_{c} - r^{\hat{b}}_{c}\delta^{a}_{d})$$

$$= R_{\hat{a}bcd} + \frac{1}{(n-1)}(r^{\hat{b}}_{[d}\delta^{a}_{c]})$$

$$= \alpha_{a[c}\delta^{b}_{d]} + \tfrac{1}{2}\alpha_{a}\alpha_{[c}\delta^{b}_{d]} + \frac{1}{(n-1)}(r^{\hat{b}}_{[d}\delta^{a}_{c]})$$

3) Put $i = a$, $j = \hat{b}$, $k = c$, and $\ell = d$, we get:

$$W_{a\hat{b}cd} = R_{a\hat{b}cd} + \frac{1}{2(n-1)}(r_{ac}g_{\hat{b}d} + r_{\hat{b}d}g_{ac} - r_{ad}g_{\hat{b}c} - r_{\hat{b}c}g_{ad}) + \frac{K(g_{\hat{b}c}g_{ad} - g_{\hat{b}d}g_{ac})}{(2n-1)(2n-2)}$$

$$= R_{a\hat{b}cd} + \frac{1}{2(n-1)}(r^{\hat{a}}_{c}\delta^{b}_{d} - r^{\hat{a}}_{d}\delta^{b}_{c})$$

$$= R_{a\hat{b}cd} + \frac{1}{(n-1)}(r^{\hat{a}}_{[c}\delta^{b}_{d]})$$

$$= -\alpha_{a[c}\delta^{b}_{d]} - \tfrac{1}{2}\alpha_{a}\alpha_{[c}\delta^{b}_{d]} + \frac{1}{(n-1)}(r^{\hat{a}}_{[c}\delta^{b}_{d]})$$

4) Put $i = a$, $j = b$, $k = \hat{c}$, and $\ell = d$, we get:

$$W_{ab\hat{c}d} = R_{ab\hat{c}d} + \frac{1}{2(n-1)}(r_{a\hat{c}}g_{bd} + r_{bd}g_{a\hat{c}} - r_{ad}g_{b\hat{c}} - r_{b\hat{c}}g_{ad}) + \frac{K(g_{b\hat{c}}g_{ad} - g_{bd}g_{a\hat{c}})}{(2n-1)(2n-2)}$$

$$= R_{ab\hat{c}d} + \frac{1}{2(n-1)}(r^{\hat{d}}_{b}\delta^{c}_{a} - r^{\hat{d}}_{a}\delta^{c}_{b})$$

$$= R_{ab\hat{c}d} + \frac{1}{(n-1)}(r^{\hat{d}}_{[b}\delta^{c}_{a]})$$

$$= \alpha_{[a|d|}\delta^c_{b]} - \alpha_{[a}\delta^h_{b]}\alpha_{[h}\delta^c_{d]} + \tfrac{1}{(n-1)}(r^{\hat{d}}_{[b}\delta^c_{a]})$$

5) Put $i = a$, $j = b$, $k = c$, and $\ell = \hat{d}$, we obtained:

$$W_{abc\hat{d}} = R_{abc\hat{d}} + \tfrac{1}{2(n-1)}(r_{ac}g_{b\hat{d}} + r_{b\hat{d}}g_{ac} - r_{a\hat{d}}g_{bc} - r_{bc}g_{a\hat{d}}) + \frac{K(g_{bc}g_{a\hat{d}} - g_{b\hat{d}}g_{ac})}{(2n-1)(2n-2)}$$
$$= R_{abc\hat{d}} + \tfrac{1}{2(n-1)}(r^{\hat{c}}_a\delta^d_b - r^{\hat{c}}_b\delta^d_a)$$
$$= \alpha_{[a|c|}\delta^d_{b]} - \alpha_{[a}\delta^h_{b]}\alpha_{[h}\delta^d_{c]} + \tfrac{1}{(n-1)}(r^{\hat{c}}_{[a}\delta^d_{b]})$$

6) Put $i = \hat{a}$, $j = \hat{b}$, $k = c$, and $\ell = d$, we have:

$$W_{\hat{a}\hat{b}cd} = R_{\hat{a}\hat{b}cd} + \tfrac{1}{2(n-1)}(r_{\hat{a}c}g_{\hat{b}d} + r_{\hat{b}d}g_{\hat{a}c} - r_{\hat{a}d}g_{\hat{b}c} - r_{\hat{b}c}g_{\hat{a}d}) + \frac{K(g_{\hat{b}c}g_{\hat{a}d} - g_{\hat{b}d}g_{\hat{a}c})}{(2n-1)(2n-2)}$$
$$= R_{\hat{a}\hat{b}cd} + \tfrac{1}{2(n-1)}(r^a_c\delta^b_d + r^b_d\delta^a_c - r^a_d\delta^b_c - r^b_c\delta^a_d) + \frac{K(\delta^b_c\delta^a_d - \delta^b_d\delta^a_c)}{(2n-1)(2n-2)}$$
$$= -2\alpha^{[a}_{[c}\delta^{b]}_{d]} + \tfrac{1}{(n-1)}(r^a_{[c}\delta^b_{d]} + r^b_{[d}\delta^a_{c]}) + \frac{K\delta^{ba}_{cd}}{(2n-1)(n-1)}$$

7) Put $i = \hat{a}$, $j = b$, $k = \hat{c}$, and $\ell = d$, we get:

$$W_{\hat{a}b\hat{c}d} = R_{\hat{a}b\hat{c}d} + \tfrac{1}{2(n-1)}(r_{\hat{a}\hat{c}}g_{bd} + r_{bd}g_{\hat{a}\hat{c}} - r_{\hat{a}d}g_{b\hat{c}} - r_{b\hat{c}}g_{\hat{a}d}) + \frac{K(g_{b\hat{c}}g_{\hat{a}d} - g_{bd}g_{\hat{a}\hat{c}})}{(2n-1)(2n-2)}$$
$$= R_{\hat{a}b\hat{c}d} - \tfrac{1}{2(n-1)}(r^a_d\delta^c_b + r^c_b\delta^a_d) + \frac{K\delta^c_b\delta^a_d}{(2n-1)(2n-2)}$$
$$= R_{\hat{a}b\hat{c}d} - \tfrac{1}{(n-1)}r^{(a}_{(d}\delta^{c)}_{b)} + \frac{K\delta^c_b\delta^a_d}{(2n-1)(2n-2)}$$
$$= A^{ad}_{bc} - \alpha^{[a}\delta^{h]}_d\alpha_{[b}\delta^c_{h]} - \tfrac{1}{(n-1)}r^{(a}_{(d}\delta^{c)}_{b)} + \frac{K\delta^c_b\delta^a_d}{(2n-1)(2n-2)}$$

8) Put $i = \hat{a}$, $j = b$, $k = c$, and $\ell = \hat{d}$, we have:

$$W_{\hat{a}bc\hat{d}} = R_{\hat{a}bc\hat{d}} + \tfrac{1}{2(n-1)}(r_{\hat{a}c}g_{b\hat{d}} + r_{b\hat{d}}g_{\hat{a}c} - r_{\hat{a}\hat{d}}g_{bc} - r_{bc}g_{\hat{a}\hat{d}}) + \frac{K(g_{bc}g_{\hat{a}\hat{d}} - g_{b\hat{d}}g_{\hat{a}c})}{(2n-1)(2n-2)}$$
$$= R_{\hat{a}bc\hat{d}} + \tfrac{1}{2(n-1)}(r^a_c\delta^d_b + r^b_d\delta^a_c) - \frac{K\delta^b_d\delta^a_c}{(2n-1)(2n-2)}$$
$$= A^{ad}_{bc} - \alpha^{[a}\delta^{h]}_c\alpha_{[h}\delta^d_{b]} + \tfrac{1}{2(n-1)}(r^a_c\delta^d_b + r^b_d\delta^a_c) - \frac{K\delta^b_d\delta^a_c}{(2n-1)(2n-2)} \; .$$

Definition 3.19. An *AH*-manifold is called a conformal flat if the Welye's tensor equal to zero.

Theorem 3.20. If M is *L.C.K*- manifold of the *PHS*-curvature tensor and is a conformal flat with $J-$invariant Ricci tensor, then M is an Einstein manifold.

Proof. suppose that M is *L.C.K*-manifold of the *PHS*-curvature tensor and conformal flat with $J-$invariant Ricci tensor.

Put $i=a, j=\hat{b}, k=\hat{c}$ and $\ell=d$, then the equation (3.5) becomes:

$$W_{a\hat{b}\hat{c}d}=R_{a\hat{b}\hat{c}d}+\tfrac{1}{2(n-1)}(r_{a\hat{c}}g_{\hat{b}d}+r_{\hat{b}d}g_{a\hat{c}}-r_{ad}g_{\hat{b}\hat{c}}-r_{\hat{b}\hat{c}}g_{ad})+\frac{K(g_{\hat{b}\hat{c}}g_{ad}-g_{\hat{b}d}g_{a\hat{c}})}{(2n-1)(2n-2)}$$

$$=R_{a\hat{b}\hat{c}d}+\tfrac{1}{2(n-1)}(r_a^c\delta_d^b+r_d^b\delta_a^c)-\frac{K\delta_d^b\delta_a^c}{(2n-1)(2n-2)}$$

By using the definition 3.19, and theorems 3.4 , 3.11, we obtained:

$$\tfrac{-c}{2}\tilde{\delta}_{ad}^{bc}+\tfrac{1}{2(n-1)}(r_a^c\delta_d^b+r_d^b\delta_a^c)-\frac{K\delta_d^b\delta_a^c}{(2n-1)(2n-2)}=0$$

Thus

$$\tfrac{-c}{2}[\delta_a^b\delta_d^c+\delta_d^b\delta_a^c]+\tfrac{1}{2(n-1)}(r_a^c\delta_d^b+r_d^b\delta_a^c)-\frac{K\delta_d^b\delta_a^c}{(2n-1)(2n-2)}=0 \qquad (3.6)$$

By contracting (3.6) by the indices (c,d) we obtained:

$r_a^b=e\delta_a^b$, where $e=\dfrac{c}{2}(n+1)(n-1)+\dfrac{K}{2(2n-1)}$.

Since Ricci tensor is $J-$invariant, so by Lemma 3.13 we get:

$r_{ij}=eg_{ij}$, where e is an Einstein constant.

Therefore M is an Einstein manifold.

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