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Convergence of the Projection Type Ishikawa Iteration Process with Errors for a Finite Family of I-Asymptotically Nonexpansive Mappings in Generalized Convex Metric Spaces

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Abstract

In this paper, we consider the strong convergence of the projection type Ishikawa iteration process to a common fixed point of a finite family of I-asymptotically nonexpansive mappings in generalized convex metric spaces. Our results of this paper improve and extend the corresponding results of Temir[4], Thianwan [6].

Keywords: A finite family of I-asymptotically nonexpansive mappings, Generalized convex metric spaces, Projection type Ishikawa iteration process

1 Introduction and preliminaries

Let (X, d) be a metric space and C be a nonempty closed convex subset of X. Let T be a self-mapping of C. T is said to be asymptotically nonexpansive if there exists $k_n \in [0, \infty)$, $lim_{n\to\infty} k_n = 0$, such that $d(T^n x, T^n y) \leq (1 + k_n)d(x, y)$ for all $x, y \in C$. Let $F(T) = \{x \in X : Tx = x\}$; if $F(T) \neq \emptyset$, then T is called asymptotically quasi-nonexpansive if there exists $k_n \in [0, \infty)$, $lim_{n\to\infty} k_n = 0$, such that $d(T^n x, p) \leq (1 + k_n)d(x, p)$ for all $x \in C$, $p \in F(T)$. Further more, it is I asymptotically quasi-nonexpansive if there

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exists sequence $\{v_n\} \subset [0, \infty)$ with $lim_{n\to\infty} v_n = 0$ such that $d(T^n x, p) \leq (1 + v_n)d(I^n x, p)$ for all $x \in C$, $p \in F(T)$, where $I : C \to C$ be asymptotically nonexpansive mappings with $\{u_n\} \subset [0, \infty)$. All of the above mappings are contractive mappings.

Remark 1.1. From above definitions it is easy to see that if $F(T)$ is nonempty, an asymptotically nonexpansive must be asymptotically quasi-nonexpansive. But the converse does not hold.

There are a number of recent results on Fixed points of asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergences of the sequence of certain iterates to a fixed point of asymptotically quasi-nonexpansive mappings were studied by Schu [2]. Takahashi [3] introduced a notion of a convex metric space which is a more general space, and each linear normed space is a special example of a convex metric space. Subsequently, Tian [5] gave some sufficient and necessary conditions for an Ishikawa iteration process of asymptotically quasi-nonexpansive mappings to converge to fixed points in convex metric spaces. Recently, Wang and Liu [7] gave some results for an Ishikawa type iteration process with errors to approximate a fixed point of two uniformly quasi-Lipschitzian mappings in generalized convex metric. Temir [4] gave some sufficiency and necessary conditions for implicit iteration process to approximate a fixed point of a finite family of I-asymptotically nonexpansive mappings in uniformly convex Banach space. Moreover, Thianwan [6] investigated the approximation of fixed point of projection type Ishikawa iteration for two asymptotically nonexpansive nonself-mappings in uniformly convex Banach space.

Inspired and motivated by the above mentioned works, in this paper, we consider the Projection type Ishikawa process with errors to approximating common fixed point for a finite family of I-asymptotically nonexpansive mappings, and obtain the strong convergence theorems for such mappings in generalized convex metric space.

We restate the following definitions and lemmas:

Definition 1.1. [3] Let (X, d) be a metric space, and $I = [0, 1]$. A mapping $w : X^2 \times I \to X$ is said to be convex structure on X, if for any $(x, y, \lambda) \in X^2 \times I$ and $u \in X$, the following inequality holds:

$$d(w(x, y, \lambda), u) \geq \lambda d(x, u) + (1 - \lambda)d(y, u).$$

If (X,d) is a metric space with a convex structure w, then (X,d) is called a convex metric space. Moreover, a nonempty subset C of X is said to be convex if $w(x,y,\lambda) \in C$, for all $(x,y,\lambda) \in C^2 \times I$.

Definition 1.2. [7] Let (X,d) be a metric space, $I=[0,1]$, and $\{a_n\},\{b_n\},\{c_n\}$ real sequences in [0,1] with $a_n+b_n+c_n=1$. A mapping $w: X^3 \times I^3 \to X$ is said to be convex structure on X, if for any $(x,y,z,a_n,b_n,c_n) \in X^3 \times I^3$ and $u \in X$, the following inequality holds:

$$d(w(x,y,z,a_n,b_n,c_n),u) \geq a_n d(x,u) + b_n d(y,u) + c_n d(z,u).$$

If (X,d) is a metric space with a convex structure w, then (X,d) is called a generalized convex metric space. Moreover, a nonempty subset C of X is said to be convex if $w(x,y,z,a_n,b_n,c_n) \in C$, for all $(x,y,z,a_n,b_n,c_n) \in C^3 \times I^3$.

Remark 1.2. It is easy to see that every generalized convex metric space is a convex metric space.

Definition 1.3. Let (X,d) be a metric space with a convex structure $w: X^3 \times I^3 \to X$. Let $T_i: X \to X, i \in \{1,...,N\}$, T_i is I_i-asymptotically quasi-nonexpansive mappings, I_i is asymptotically nonexpansive. Then an iterative scheme is the sequences of mappings $\{x_n\}$ defined by,

$$\begin{array}{rcl} x_{n+1} &=& w(y_n, I_{i(n)}^{k(n)} y_n, u_n, a_n, b_n, c_n), \\ y_n &=& w(x_n, T_{i(n)}^{k(n)} x_n, v_n, a'_n, b'_n, c'_n), \end{array} \quad n \geq 1, \tag{1.1}$$

where $\{a_n\},\{b_n\},\{c_n\},\{a'_n\},\{b'_n\},\{c'_n\}$ are real sequences in $(0,1)$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, $n = (k(n)-1)N + i(n), i(n) \in \{1,...,N\}$. $\{u_n\},\{v_n\}$ are two two sequences in X satisfying the following condition: For any non-negative integers $n,m, 0 \geq n < m$, if $\delta(A_{nm}) > 0$, then

$$max_{n \geq i, j \geq m, 1 \geq k \geq N}\{d(x,y) : x \in \{u_i, v_i\}, y \in \{x_j, y_j, I_k y_j, T_k x_j, u_j, v_j\}\} < \delta(A_{nm}), \tag{1.2}$$

where $A_{nm} = \{x_i, y_i, I_k y_i, T_k x_i, u_i, v_i : n \geq i \geq m, 1 \geq k \geq N\}$, $\delta(A_{nm}) = sup_{x,y \in A_{nm}} d(x,y)$. Then $\{x_n\}$ is called the Projection type Ishikawa iteration process with errors for a finite family of I-asymptotically nonexpansive mappings T_i.

Note that the iteration processes considered in [6] can be obtained from the above process as special cases by suitably choosing the space and the mappings.

Lemma 1.1 [1]. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\mu_n\}$ be four nonnegative real sequences satisfying $\alpha_{n+1} \leq (1+\gamma_n)(1+\mu_n)\alpha_n + \beta_n$, for all $n \geq 1$. If $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $lim_{n \to \infty} \alpha_n$ exists.

2 Main Results

Lemma 2.1. Let X be a generalized convex metric space, C be a nonempty closed convex subset of X, $\{T_i : i \in \{1, 2, ..., N\}\} : C \to C$ be N I_i-asymptotically quasi-nonexpansive mappings with sequences $\{v_{in}\} \subset [0, \infty)$ such that $\Sigma_{n=1}^{\infty} v_{in} < \infty$ and $\{I_i : i \in \{1, \ldots, N\}\} : C \to C$ be N asymptotically nonexpansive mappings with $\{u_{in}\} \subset [0, \infty)$ such that $\Sigma_{n=1}^{\infty} u_{in} < \infty$ and $F = \cap_{i=1}^{N} F(T_i) \cap F(I_i)$. The Projection type Ishikawa iteration sequence $\{x_n\}$ is generated by (1.1), $\{u_n\}, \{v_n\}$ satisfying (1.2). If $F \neq \emptyset$ and $\Sigma_{n=1}^{\infty} c_n + c'_n < \infty$, then (1) F is closed, (2) $lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F$.

Proof. (1). Let $\{\xi_n\} \subset F$ be such that $\xi_n \to x$ as $n \to \infty$. In addition, for $\forall i \in \{1, 2, ..., N\}$,

$$d(T_i x, x) = d(T_i x, T_i \xi_n) + d(\xi_n, x) \leq (1 + u_{i1} v_{i1}) d(\xi_n, x)$$

This implies that $T_i x = x$. By the same way, we can get that $x \in F(I_i)$. So, x is a common fixed point of T_i and I_i. Thus F is a closed set.

(2). For any $p \in F = \cap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$.

$$\begin{aligned}
d(x_{n+1}, p) &= d(w(y_n, I_{i(n)}^{k(n)} y_n, u_n, a_n, b_n, c_n), p) \\
&\leq a_n d(y_n, p) + b_n d(I_{i(n)}^{k(n)} y_n, p) + c_n d(u_n, p) \\
&\leq (1 - b_n) d(y_n, p) - c_n d(y_n, p) + c_n d(u_n, p) + b_n d(I_{i(n)}^{k(n)} y_n, p) \\
&\leq (1 - b_n) d(y_n, p) + c_n d(y_n, u_n) + b_n (1 + u_{ik}) d(y_n, p) \\
&\leq (1 + b_n u_{ik}) d(y_n, p) + c_n d(y_n, u_n). \qquad (2.1) \\
d(y_n, p) &= d(w(x_n, T_{i(n)}^{k(n)} x_n, v_n, a'_n, b'_n, c'_n), p) \\
&\leq a'_n d(x_n, p) + b'_n d(T_{i(n)}^{k(n)} x_n, p) + c'_n d(v_n, p) \\
&\leq (1 - b'_n) d(x_n, p) - c'_n d(x_n, p) + c'_n d(v_n, p) + b'_n d(T_{i(n)}^{k(n)} x_n, p) \\
&\leq (1 - b'_n) d(x_n, p) + c'_n d(x_n, v_n) + b'_n (1 + u_{ik})(1 + v_{ik}) d(x_n, p) \\
&\leq (1 + b'_n (u_{ik} + v_{ik} + u_{ik} v_{ik})) d(x_n, p) + c'_n d(x_n, v_n). \qquad (2.2)
\end{aligned}$$

So,

$$\begin{aligned}
d(x_{n+1}, p) &= (1 + b_n u_{ik})[(1 + \beta_n) d(x_n, p) + c'_n d(x_n, v_n)] + c_n d(y_n, u_n) \\
&\leq (1 + \alpha_n)(1 + \beta_n) d(x_n, p) + (1 + \alpha_n) c'_n d(x_n, v_n) + c_n d(y_n, u_n) \\
&\leq (1 + \alpha_n)(1 + \beta_n) d(x_n, p) + \gamma_n M. \qquad (2.3)
\end{aligned}$$

where $\alpha_n = b_n u_{ik}, \beta_n = b'_n(u_{ik} + v_{ik} + u_{ik}v_{ik}), \gamma_n = c_n + c'_n$, and $M = (1 + u_{ik})(d(x_n, v_n) + d(y_n, u_n))$,

Notice that $\Sigma_{k=1}^{\infty} u_{ik} < \infty, \Sigma_{k=1}^{\infty} v_{ik} < \infty$ for all $i \in \{1, 2...N\}$, $\Sigma_{n=1}^{\infty} c_n + c'_n < \infty$, for all $n \in \mathcal{N}$, and $\{u_n\}, \{v_n\}$ satisfying (1.2), we have that $lim_{n\to\infty}\alpha_n = lim_{n\to\infty}\beta_n = lim_{n\to\infty}\gamma_n = 0$.

By Lemma 1.1, $lim_{n\to\infty}d(x_n, p)$ exists for each $p \in F$. The proof is completed.

Theorem 2.2. Let X be a generalized convex metric space, $C, \{T_i\}, \{I_i\}, \{x_n\}$ be same as in Lemma 2.1 and $F \neq \emptyset$. The Projection type Ishikawa iteration sequence $\{x_n\}$ converges strongly to a common fixed point in F if and only if $liminf_{n\to\infty}d(x_n, F) = 0$, where $d(x, F) = inf\{d(x, p) : p \in F\}$.

Proof. The necessity of the conditions is obvious. Thus, we will only prove the sufficiency.

By Lemma 2.1, $lim_{n\to\infty}d(x_n, p)$ exists for each $p \in F$. Hence (2.3) implies that $d(x_n, F) \leq (1 + \alpha_n)(1 + \beta_n)d(x_n, F) + \gamma_n M$. From Lemma 1.1 $lim_{n\to\infty}d(x_n, F)$ exists and by the hypothesis $liminf_{n\to\infty}d(x_n, F) = 0$. We have $lim_{n\to\infty}d(x_n, F) = 0$.

Next we show that $\{x_n\}$ is a cauchy sequence.

Let $\varepsilon > 0$, since $lim_{n\to\infty}d(x_n, F) = 0$, there exist natural number N_1 such that when $n \geq N_1$, $d(x_n, F) < \frac{\varepsilon}{3}$. Thus, there exists $x^* \in F$ such that for above ε there exists positive integer $N_2 \geq N_1$ such that as $n \geq N_2$, $d(x_n, x^*) < \frac{\varepsilon}{2}$. Now for arbitrary $n, m \geq N_2$, consider $d(x_n, x_m) \leq d(x_n, x^*) + d(x_m, x^*) < \varepsilon$. This implies that $\{x_n\}$ is a cauchy sequence in C, therefor it converges to a point, say $p \in C$. And $lim_{n\to\infty}d(x_n, F) = 0$ gives that $d(p, F) = 0$. By Lemma 2.1, we know F is closed. Thus $p \in F$. The proof is completed.

Corollary 2.3. Let C be a nonempty closed convex subset of a Banach space X, $\{T_i\}, \{I_i\}$ be same as in Lemma 2.1. Let $F = \cap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. For any given $x_1 \in C$, $\{x_n\}$ is a Projection type Ishikawa iteration process with errors defined by

$$\begin{array}{rcl} x_{n+1} & = & a_n y_n + b_n I_{i(n)}^{k(n)} y_n + c_n u_n, \\ y_n & = & a'_n x_n + b'_n T_{i(n)}^{k(n)} x_n + c'_n v_n, \end{array} \quad n \geq 1, \tag{2.4}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0, 1)$ with $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. If

$\Sigma_{n=1}^{\infty} c_n + c_n' < \infty$, then $\{x_n\}$ converges strongly to a common fixed point in F if and only if $liminf_{n\to\infty} d(x_n, F) = 0$.

Theorem 2.4. Let X be a generalized convex metric space, C be a nonempty closed convex subset of X, $\{T_i : i \in \{1, 2, ..., N\}\} : C \to C$ be N I_i-asymptotically quasi-nonexpansive mappings with sequences $\{v_{in}\} \subset [0, \infty)$, $\{I_i : i \in \{1, \ldots, N\}\} : C \to C$ be N asymptotically nonexpansive mappings with $\{u_{in}\} \subset [0, \infty)$(without the conditions $\Sigma_{n=1}^{\infty} u_{in} < \infty$ and $\Sigma_{n=1}^{\infty} v_{in} < \infty$) and $F = \cap_{i=1}^{N} F(T_i) \cap F(I_i)$. The Projection type Ishikawa iteration sequence $\{x_n\}$ is generated by (1.1), $\{u_n\}, \{v_n\}$ satisfying (1.2). If $F \neq \emptyset$ and $\Sigma_{n=1}^{\infty} a_n + b_n + c_n < \infty$, then $\{x_n\}$ converges strongly to a common fixed point in F if and only if $liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. For any $p \in F$.

$$\begin{aligned} d(x_{n+1}, p) &= d(w(y_n, I_{i(n)}^{k(n)} y_n, u_n, a_n, b_n, c_n), p) \\ &\leq a_n d(y_n, p) + b_n d(I_{i(n)}^{k(n)} y_n, p) + c_n d(u_n, p) \\ &\leq a_n d(y_n, p) + b_n (1 + u_{ik}) d(y_n, u_n) + c_n d(u_n, p). \qquad (2.5) \\ d(y_n, p) &= d(w(x_n, T_{i(n)}^{k(n)} x_n, v_n, a_n', b_n', c_n'), p) \\ &\leq a_n' d(x_n, p) + b_n' d(T_{i(n)}^{k(n)} x_n, p) + c_n' d(v_n, p) \\ &\leq a_n' d(x_n, p) + b_n' (1 + u_{ik})(1 + v_i k) d(x_n, p) + c_n' d(v_n, p). \qquad (2.6) \end{aligned}$$

we have,

$$\begin{aligned} d(x_{n+1}, p) &= (a_n + b_n(1 + u_{ik}))[\gamma_n d(x_n, p) + c_n' d(v_n, p)] + c_n d(u_n, p) \\ &\leq (1 + \alpha_n) d(x_n, p) + \alpha_n M. \qquad (2.7) \end{aligned}$$

where $\alpha_n = a_n + b_n + c_n$, $\gamma_n = a_n' + b_n'(1 + u_{ik})(1 + v_{ik})$, and $M = (1 + u_{ik}) c_n' (d(v_n, p) + d(u_n, p))$.

Since $\{u_n\}, \{v_n\}$ satisfying (1.2), $\Sigma_{n=1}^{\infty} a_n + b_n + c_n < \infty$, for all $n \in \mathcal{N}$, thus $lim_{n\to\infty} \alpha_n = 0$. By Lemma 1.1, $lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Hence, Theorem 2.4 can be proven by Theorem 2.2.

Corollary 2.5. Let C be a nonempty closed convex subset of a Banach space X, $\{T_i\}, \{I_i\}$ be same as in Theorem 2.4 and $F \neq \emptyset$. $\{x_n\}$ is a Projection type Ishikawa iteration process with errors defined by (2.4). If $\Sigma_{n=1}^{\infty} a_n + b_n + c_n < \infty$, then $\{x_n\}$ converges strongly to a common fixed point in F if and only if $liminf_{n\to\infty} d(x_n, F) = 0$

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