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On Graphoidal Covers of Bicyclic Graphs *

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Abstract

A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ and every edge of G is in exactly one path in ψ. The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$ or η. Also, If every member in a graphoidal cover is an open path then it is called an acyclic graphoidal cover. The minimum cardinality of an acyclic graphoidal cover of G is called the acyclic graphoidal covering number of G and is denoted by $\eta_a(G)$ or η_a. Here we find minimum graphoidal covering number and minimum acyclic graphoidal covering number of bicyclic graphs.

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1 Introduction

A graph is a pair $G = (V, E)$, where V is the set of vertices and E is the set of edges. Here, we consider only nontrivial, finite, connected and simple graphs. The order and size of G are denoted by p and q respectively. The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar [3] and the concept of acyclic graphoidal cover was introduced by Arumugram and Suresh Suseela [2]. The reader may refer [1] and [4] for the terms not defined here.

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2 Preliminaries

Definition 2.1. *A connected $(p, p+1)-$graph G is called a bicyclic graph.*

Definition 2.2. *A* **one-point union of two cycles** *is a simple graph obtained from two cycles, say C_l and C_m where $l, m \geq 3$, by identifying one and the same vertex from both cycles. Without loss of generality, we may assume the l-cycle to be $u_0 u_1 \ldots u_{l-1} u_0$ and the m-cycle to be $u_0 u_l u_{l+1} \ldots u_{m+l-2} u_0$. We denote this graph by $U(l; m)$.*

Definition 2.3. *A* **long dumbbell graph** *is a simple graph obtained by joining two cycles C_l and C_m, $l, m \geq 3$, with a path of length $i, i \geq 1$. Without loss of generality, we may assume $C_l = u_0 u_1 \ldots u_{l-1} u_0$, $P_i = u_{l-1} u_l \ldots u_{l+i-1}$ and $C_m = u_{l+i-1} u_{l+i} \ldots u_{l+m+i-2} u_{l+i-1}$. We denote this graph by $D(l, m; i)$.*

Definition 2.4. *A* **cycle with a long chord** *is a simple graph obtained from an m-cycle, $m \geq 4$, by adding a chord of length l where $l \geq 1$. Let the m-cycle be $u_0 u_1 \ldots u_{m-1} u_0$. Without loss of generality, we may assume the chord joins u_0 with u_i, where $2 \leq i \leq m-2$. That is, $u_0 u_m u_{m+1} \ldots u_{l+m-2} u_i$ is the chord. We denote this graph by $C_m(i; l)$.*

Definition 2.5. *[3] A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions:*
(i) Every path in ψ has at least two vertices.
(ii) Every vertex of G is an internal vertex of at most one path in ψ.
(iii) Every edge of G is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$.

Definition 2.6. *[2] An acyclic graphoidal cover of G is a graphoidal cover ψ of G such that every element of ψ is a path in G. The minimum cardinality of an acyclic graphoidal cover of G is called the acyclic graphoidal covering number of G and is denoted by $\eta_a(G)$ or η_a.*

Let ψ be a collection of internally edge disjoint paths in G. A vertex of G is said to be an internal vertex of ψ if it is an internal vertex of some path in ψ, otherwise it is called an external vertex of ψ.

Theorem 2.7. *[6] For any graphoidal cover ψ of a graph G, let t_ψ denote the number of external vertices of ψ and let $t = min\ t_\psi$, where the minimum is taken over all graphoidal covers ψ of G. Then, $\eta(G) = q - p + t$.*

Remark: The above result also holds for acyclic graphoidal covers of a graph i.e. $\eta_a(G) = q - p + t$. [2]

Theorem 2.8. *[6] Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G and let j be the number of vertices of degree greater than 2 on C. Then*

$$\eta(G) = \begin{cases} 1 & \text{if } j = 0; \\ n+1 & \text{if } j = 1 \text{ and } deg\ v = 3, \text{ where } v \text{ is the unique vertex of} \\ & \text{degree greater than 2 on } C \\ n & \text{otherwise.} \end{cases}$$

Theorem 2.9. *[5] Let T be a tree with n pendant vertices. Then*

$$\eta(T) = n - 1.$$

Theorem 2.10. *[2] Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G and let j be the number of vertices of degree greater than 2 on C. Then*

$$\eta_a(G) = \begin{cases} 2 & \text{if } j = 0; \\ n+1 & \text{if } j = 1; \\ n & \text{otherwise.} \end{cases}$$

3 Main Results

3.1 Graphoidal Covering Number

Theorem 3.1. *Let G be a bicyclic graph with n pendant vertices. Also let $U(l;m)$ be the unique bicycle in G and let j be the number of vertices of degree greater than 2 on $U(l;m)$. Then*

$$\eta(G) = \begin{cases} 2 & \text{if } G = U(l;m); \\ n+2 & \text{if either } j = 1 \text{ and } deg\ u_0 = 5 \text{ or } j = 2,\ deg\ u_0 = 4 \text{ and} \\ & \text{exactly one vertex, say } v, \text{ in } U(l;m) \text{ is of degree 3;} \\ n+1 & \text{otherwise.} \end{cases}$$

Proof. Case(i). Let $C_l = u_0u_1 \dots u_{l-1}u_0$, $C_m = u_0u_lu_{l+1} \dots u_{m+l-2}u_0$ then $\psi = \{C_l, C_m\}$ is a graphoidal cover of G with u_0 as the external vertex for both C_l and C_m. Hence, $\eta(G) = q - p + 1 = 2$.

Case(ii): Subcase(a). When $j = 1$ and *deg* $u_0 = 5$. Suppose C_m is the cycle in $U(l;m)$ which have no vertex of *deg* > 2 except u_0. Then $G_1 = G - C_m$ is a unicyclic graph with one vertex of *deg* 3 in the cycle so that $\eta(G_1) = n+1$. Let ψ_1 be a minimum graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is a graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta(G) \leq n + 2$. Again, for any graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $U(l;m)$, say u_0, are external vertex in ψ so that $t \geq n+1$. Hence, $\eta(G) = q - p + t \geq n + 1 + 1 = n + 2$.

Subcase(b). When $j = 2$, $deg\ u_0 = 4$ and $deg\ v = 3$ for a unique vertex v in G. Suppose v lies in C_m. Then $G_1 = G - C_l$ is a unicyclic graph with $deg\ v = 3$ in the cycle C_m so that $\eta(G_1) = n+1$. Let ψ_1 be a minimum graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_l$ is a graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n+2$. Hence, $\eta(G) \leq n + 2$. Again, for any graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $U(l;m)$, say either u_0 or v, are external vertex in ψ so that $t \geq n+1$. Hence, $\eta(G) = q - p + t \geq n + 1 + 1 = n + 2$.

Case(iii): Subcase(a). When $j = 1$ and $deg\ u_0 \geq 6$. Let C_l and C_m be the two cycles. Then $T = G - C_l - C_m$ is a tree with n pendant vertices so that $\eta(T) = n - 1$, with ψ_1 as a minimum graphoidal cover of T. Then $\psi = \psi_1 \cup C_l \cup C_m$ is a minimum graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta(G) = n - 1 + 1 + 1 = n + 1$.

Subcase(b). When $j \geq 3$. Let v, w be the vertices in $U(l;m)$ which have $deg > 2$ other than u_0. Suppose $v \in C_l$ and $w \in C_m$. Then u_0 is in $v - w$ section of $U(l;m)$ such that all internal vertices except u_0 is of deg 2. Let P denote this $v - w$ section. Then $\eta(P) = \{v - u_0, u_0 - w\} = 1 + 1 = 2$. Also $T = G - P$ is a tree with n pendant vertices such that $\eta(T) = n - 1$. Let ψ_1 be a minimum graphoidal cover of T. Then $\psi = \psi_1 \cup P$ is a minimum graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta(G) = n - 1 + 2 = n + 1$. Next, suppose $v, w \in C_l$. Then we choose $v - w$ section such that u_0 is not in $v - w$ section. Let P denote this $v - w$ section of G. Then each of its internal vertices is of deg 2 and $\eta(P) = 1$. Let ψ_1 be the minimum graphoidal cover of P. Then $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta(G_1) = n$, with ψ_2 as a minimum graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2$ is a minimum graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta(G) = n + 1$. □

Theorem 3.2. *Let G be a bicyclic graph with n pendant vertices. Also let $D(l,m;i)$ be the unique bicycle in G and let j be the number of vertices of degree greater than 2 on cycles in $D(l,m;i)$. Then*

$$\eta(G) = \begin{cases} 3 & \text{if } G = D(l,m;i); \\ n+2 & \text{if } j = 2,\ deg\ u_{l-1} \geq 4 \text{ in } C_m \text{ and } deg\ u_{l+i-1} = 3 \text{ in } C_l; \\ & \text{or } j \geq 3 \text{ and vertices of } deg \geq 3 \text{ other than } u_{l-1} \text{ and } u_{l+i-1} \\ & \text{are in one of the cycles } C_m \text{ or } C_l; \\ n+1 & \text{otherwise.} \end{cases}$$

Proof. Case(i). Let $C_l = u_0u_1 \dots u_{l-1}u_0$, $P_i = u_{l-1}u_l \dots u_{l+i-1}$ and $C_m = u_{l+i-1}u_{l+i} \dots u_{l+m+i-2}u_{l+i-1}$. Then $\psi = \{C_l, P_i, C_m\}$ is a graphoidal cover of G with u_{l-1} and u_{l+i-1} as its external vertices. Hence, $\eta(G) = q - p + 2 = 3$.

Case(ii): Subcase(a). When $j = 2$, $deg\ u_{l-1} \geq 4$ in C_m and $deg\ u_{l+i-1} = 3$ in C_l. Here, $T = G - C_m - C_l$ is a tree with $n + 1$ pendant vertices so that $\eta(T) = n$, with ψ_1 as a minimum graphoidal cover of T. Then $\psi = \psi_1 \cup C_l \cup C_m$ is a graphoidal cover of G and $|\psi| = |\psi_1| + 1 + 1 = n + 2$. Hence, $\eta(G) \leq n + 2$. Again, for any graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $D(l, m; i)$, say u_{l+i-1}, are external vertex in ψ so that $t \geq n + 1$. Hence, $\eta(G) = q - p + t \geq n + 1 + 1 = n + 2$. Subcase(b). When $j \geq 3$ and vertices of $deg \geq 3$ other than u_{l-1} and u_{l+i-1} are in one of the cycles C_m or C_l. Suppose v lie in C_l. Then $G_1 = G - C_m$ is a unicyclic graph with $n+1$ pendant vertices so that $\eta(G_1) = n + 1$. Let ψ_1 be a minimum graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is a graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2$. Hence, $\eta(G) \leq n + 2$. Again, for any graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $D(l, m; i)$, say u_{l+i-1}, are external vertex in ψ so that $t \geq n + 1$. Hence, $\eta(G) = q - p + t \geq n + 1 + 1 = n + 2$.

Case(iii): Subcase(a).When $j = 2$. Then $T = G - C_l - C_m$ is a tree with n pendant vertices so that $\eta(T) = n - 1$, with ψ_1 as a minimum graphoidal cover of T. Then $\psi = \psi_1 \cup C_l \cup C_m$ is a minimum graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta(G) = n - 1 + 1 + 1 = n + 1$.

Subcase(b). When $j \geq 3$. Let v, w be the vertices in $D(l, m; i)$ of $deg > 2$. Suppose v lie in C_m and w lies in C_l. Let P denote the $v - u_{l-1}$ section of C_m such that each of its internal vertices is of deg 2. Then $\eta(P) = 1$. Let ψ_1 be the minimum graphoidal cover of P. Then $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta(G_1) = n$, with ψ_2 as a minimum graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2$ is a minimum graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta(G) = n + 1$. □

Theorem 3.3. *Let G be a bicyclic graph with n pendant vertices. Also let $C_m(i; l)$ be the unique bicycle in G and let j be the number of vertices of degree greater than 2 on cycles in $C_m(i; l)$. Then*

$$\eta(G) = \begin{cases} 2 & \text{if } G = C_m(i; l); \\ n + 2 & \text{if } j = 3 \text{ and } deg\ v = 3, \text{ where } v \text{ is the unique vertex in} \\ & C_m(i; l) \text{ other than } u_0 \text{ and } u_i; \\ n + 1 & \text{otherwise.} \end{cases}$$

Proof. Case(i). Let $C_m = u_0u_1 \dots u_{m-1}u_0$, with $m \geq 4$ be the cycle and $P_l = u_0u_mu_{m+1} \dots u_{l+m-2}u_i, 2 \leq i \leq m - 2$, be the chord in $C_m(i; l)$. Then

$\psi = \{C_m, P_l\}$ is a minimum graphoidal cover of G such that any one vertex in C_m can be taken as an external vertex. Hence, $\eta(G) = q - p + 1 = 2$.

Case(ii). Let $P_l = u_0u_mu_{m+1}\dots u_{l+m-2}u_i, 2 \leq i \leq m-2$, be the chord in $C_m(i;l)$ such that $\eta(P_l) = 1$. Then $G_1 = G - P_l$ is a unicyclic graph with n pendant vertices so that $\eta(G_1) = n+1$. Let ψ_1 be a minimum graphoidal cover of G_1. Then $\psi = \psi_1 \cup P_l$ is a graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n+2$. Hence, $\eta(G) \leq n+2$. Again, for any graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $C_m(i;l)$, say v, are external vertex in ψ so that $t \geq n+1$. Hence, $\eta(G) = q - p + t \geq n + 1 + 1 = n + 2$.

Case(iii): Subcase(a). When $j = 2$, $deg\ u_0 = 3$ and $deg\ u_i \geq 4$. Let C_m be the cycle with u_i as an external vertex and P_l be the chord in G along with its n pendant vertices. Then $T = G - C_m$ is a tree with $n+1$ pendant vertices so that $\eta(T) = n$, with ψ_1 as a minimum graphoidal cover of T. Now, $\psi = \psi_1 \cup C_m$ is a minimum graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta(G) = n + 1$.

Subcase(b). When $j \geq 3$. Let $P_l = u_0u_mu_{m+1}\dots u_{l+m-2}u_i, 2 \leq i \leq m-2$, be the chord in G such that $\eta(P_l) = 1$. Then $G_1 = G - P_l$ is a unicyclic graph with n pendant vertices so that $\eta(G_1) = n$. Let ψ_1 be a minimum graphoidal cover of G_1. Then $\psi = \psi_1 \cup P_l$ is a minimum graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta(G) = n + 1$. □

3.2 Acyclic Graphoidal Covering Number

Theorem 3.4. *Let G be a bicyclic graph with n pendant vertices. Also let $U(l;m)$ be the unique bicycle in G and let j be the number of vertices of degree greater than 2 on $U(l;m)$. Then*

$$\eta_a(G) = \begin{cases} 3 & \text{if } G = U(l;m); \\ n+3 & \text{if } j = 1 \text{ and } deg\ u_0 \geq 4; \\ n+2 & \text{if } j \geq 2 \text{ and } deg\ v_i \geq 3 \text{ so that either } v_i,\ 0 \leq i \leq l-1, \\ & \text{are on } C_l \text{ or } v_i,\ 0 \leq i \leq m+l-2, \text{ are on } C_m; \\ n+1 & \text{otherwise.} \end{cases}$$

Proof. Case(i). When $G = U(l;m)$.
Let $P_1 = \{u_1, u_2, \dots, u_{l-1}, u_0, u_l, u_{l+1}, \dots, u_{m+l-2}\}$, $P_2 = \{u_0, u_{m+l-2}\}$ and $P_3 = \{u_0, u_1\}$ then $\psi = \{P_1, P_2, P_3\}$ is an acyclic graphoidal cover of G with u_1 and u_{m+l-2} as its external vertices. Hence $\eta_a(G) = 3$.

Case(ii). When $j = 1$ and $deg\ u_0 \geq 4$. Let e be an edge on either cycle incident at u_0 then $G_1 = G - e$ is a unicyclic graph with $n + 1$ pendant vertices so that $\eta_a(G_1) = n + 1 + 1 = n + 2$, with ψ_1 as a minimum acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup e$ is an acyclic graphoidal cover of

G and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_a(G) \leq n + 3$. Again, for any acyclic graphoidal cover ψ of G, the n pendant vertices of G and at least two vertices in $U(l;m)$ are external vertices in ψ so that $t \geq n + 2$. Hence, $\eta_a(G) = q - p + t \geq n + 2 + 1 = n + 3$.

Case(iii). When $j \geq 2$ and $deg\ v_i \geq 3$ so that $v_i,\ 0 \leq i \leq l-1$, are on C_l. Let P be the path between two vertices which have degree greater than 2 in C_l. Then each of its internal vertices is of deg 2 and $\eta_a(P) = 1$. Also $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_a(G_1) = n+1$, with ψ_1 as a minimum acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is an acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 1 + 1 = n + 2$. Hence, $\eta_a(G) \leq n + 2$. Again, for any acyclic graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $U(l;m)$ are external vertex in ψ so that $t \geq n + 1$. Hence, $\eta_a(G) = q - p + t \geq n + 1 + 1 = n + 2$.

Case(iv). When $j \geq 3$ and $deg\ u_i \geq 3,\ 0 \leq i \leq m + l - 2$, in C_l and C_m. Let P be the path between two vertices which have degree greater than 2 either in C_l or C_m. Then each of its internal vertices is of deg 2 and $\eta_a(P) = 1$. Also $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_a(G_1) = n$, with ψ_1 as a minimum acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is a minimum acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_a(G) = n + 1$ □

Theorem 3.5. *Let G be a bicyclic graph with n pendant vertices. Also let $D(l,m;i)$ be the unique bicycle in G and let j be the number of vertices of degree greater than 2 on $D(l,m;i)$. Then*

$$\eta_a(G) = \begin{cases} 3 & \text{if } G = D(l,m;i); \\ n+3 & \text{if } j = 2 \text{ and either } deg\ u_{l-1} \geq 3 \text{ or } deg\ u_{l+i-1} \geq 3; \\ n+2 & \text{if } j \geq 2 \text{ and } deg\ v_i \geq 3, \text{ so that either } v_i,\ 0 \leq i \leq l-1, \\ & \text{are on } C_l \text{ or } v_i,\ l+i-1 \leq i \leq l+m+i-2, \text{ are on } C_m; \\ n+1 & \text{otherwise.} \end{cases}$$

Proof. Case(i). When $G = D(l,m;i)$.
Let $P_1 = \{u_0, u_1, \ldots, u_{l-1}, u_l, \ldots, u_{l+i-1}, u_{l+i}, \ldots, u_{l+m+i-2}\}$, $P_2 = \{u_{l-1}, u_0\}$ and $P_3 = \{u_{l+i-1}, u_{l+m+i-2}\}$ then $\psi = \{P_1, P_2, P_3\}$ is an acyclic graphoidal cover of G with u_{l-1} and $u_{l+m+i-2}$ as its external vertex. Hence $\eta_a(G) = 3$.

Case(ii). When $j = 2$ and either $deg\ u_{l-1} \geq 3$ or $deg\ u_{l+i-1} \geq 3$. Let e be an edge on either cycle incident at u_{l-1} or u_{l+i-1} then $G_1 = G - e$ is a unicyclic graph with $n+1$ pendant vertices so that $\eta_a(G_1) = n+1+1 = n+2$, with ψ_1 as a minimum acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup e$ is an acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 3$. Hence, $\eta_a(G) \leq n + 3$. Again, for any acyclic graphoidal cover ψ of G, the n pendant vertices of G and at least two vertices in $U(l;m)$ are external vertices in ψ so that $t \geq n+2$. Hence, $\eta_a(G) = q - p + t \geq n + 2 + 1 = n + 3$.

Case(iii). When $j \geq 2$ and $deg\ v_i \geq 3$, so that v_i, $0 \leq i \leq l-1$, are on C_l. Let P be the path between two vertices which have degree greater than 2 in C_l. Then each of its internal vertices is of deg 2 and $\eta_a(P) = 1$. Also $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_a(G_1) = n+1$, with ψ_1 as a minimum acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is an acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 1 + 1 = n + 2$. Hence, $\eta_a(G) \leq n + 2$. Again, for any acyclic graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $D(l, m; i)$ are external vertex in ψ so that $t \geq n + 1$. Hence, $\eta_a(G) = q - p + t \geq n + 1 + 1 = n + 2$.

Case(iv). When $j \geq 3$. Let P be the path between two vertices which have degree greater than 2 in $D(l, m; i)$. Then each of its internal vertices is of deg 2 and $\eta_a(P) = 1$. Also $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_a(G_1) = n$. Let ψ_1 be a minimum acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is a minimum acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_a(G) = n + 1$. □

Theorem 3.6. *Let G be a bicyclic graph with n pendant vertices. Also let $C_m(i; l)$ be the unique bicycle in G and let j be the number of vertices of degree greater than 2 on $C_m(i; l)$. Then*

$$\eta_a(G) = \begin{cases} 2 & \text{if } G = C_m(i; l); \\ n+2 & \text{if } j = 3, \text{ degree of both } u_0, u_i \text{ is 3 and } deg\ v \geq 3, \\ & \text{where } v \text{ is the unique vertex in } C_m; \\ n+1 & \text{otherwise.} \end{cases}$$

Proof. Case(i). When $G = C_m(i; l)$. Let $P_1 = \{u_0, u_1, \ldots, u_{m-1}\}$ and $P_2 = \{u_{m-1}, u_0, u_m, u_{m+1}, \ldots, u_{l+m-2}, u_i\}$ where $2 \leq i \leq m-2$ then $\psi = \{P_1, P_2\}$ is an acyclic graphoidal cover of G with u_{m-1} as its external vertex. Hence $\eta_a(G) = 2$.

Case(ii). When $j = 3$, degree of both u_0, u_i is 3 and $deg\ v \geq 3$, where v is the unique vertex in C_m. Let P be the path between two vertices which have degree greater than 2 in $C_m(i; l)$. Then each of its internal vertices is of deg 2 and $\eta_a(P) = 1$. Also $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_a(G_1) = n + 1$, with ψ_1 as a minimum acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is an acyclic graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 1 + 1 = n + 2$. Hence, $\eta_a(G) \leq n + 2$. Again, for any acyclic graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $C_m(i; l)$ are external vertex in ψ so that $t \geq n + 1$. Hence, $\eta_a(G) = q - p + t \geq n + 1 + 1 = n + 2$.

Case(iii). When $j \geq 2$. Let P be the path between two vertices which have degree greater than 2 in $C_m(i; l)$. Then each of its internal vertices is of deg 2 and $\eta_a(P) = 1$. Also $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_a(G_1) = n$. Let ψ_1 be a minimum acyclic graphoidal cover of G_1. Then $\psi = \psi_1 \cup P$ is a minimum acyclic graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path in ψ. Hence, $\eta_a(G) = n + 1$. □

References

[1] S. Arumugam, B. D. Acharya and E. Sampathkumar, Graphoidal covers of a graph: a creative review, in *Proc. National Workshop on Graph Theory and its applications*, Manonmaniam Sundaranar University, Tirunelveli, Tata McGraw-Hill, New Delhi, (1997), 1-28.

[2] S. Arumugam and J. Suresh Suseela, Acyclic graphoidal covers and path partitions in a graph, *Discrete Math.* **190** (1998), 67-77.

[3] B. D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J. Pure Appl. Math.* **18** (10) (1987), 882-890.

[4] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.

[5] C. Pakkim and S. Arumugam, On the graphoidal covering number of a graph, *Indian J. Pure Appl. Math.* **20** (4) (1989), 330-333.

[6] C. Pakkim and S. Arumugam, The graphoidal covering number of unicyclic graphs, *Indian J. Pure Appl. Math.* **23** (2) (1992), 141-143.

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