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Coincidence and Common Fixed Points of Hybrid Mappings

Z. M. I. Sajath

Department of Mathematics
Department of Information Technology
Higher College of Technology
Muscat, Sultanate of Oman
misajath@Yahoo.com

P. Vijayaraju

Department of Mathematics
Anna University, Chennai-600 025, India
vijay@annauniv.edu

Abstract

We define a new weak contractive condition for a hybrid pair of single and multi-valued mappings and prove the existence of coincidence and common fixed points. Our results generalize various known results in the literature.

1 Introduction and preliminaries

Sessa [7]introduced the notion of weakly commuting maps in metric spaces. Jungck[6] coined the term of compatible mappings and generalized the concept of commutativity. Aamir and El Moutawakil[5]introduced the concept of E.A property and generalized noncompatible mappings. YichengLiu et al[2]defined the (EA)property for hybrid pair of single and multi-valued mappings and proved some common fixed point. In recent years several authors used these concepts to establish the existence of coincidence and common fixed points for various classes of mappings. Ismat Beg and Mujahid Abbas [1]proved the existence of coincidence and common fixed points for a pair of mappings satisfying weak contractive conditions.

The aim of this paper is to define a generalized weak contractive condition and give some new coincidence and fixed point theorems under hybrid contractive conditions. Further we extend the results regarding the coincidence and common fixed points of two mappings, one is contractive with respect to other to a hybrid pair of mappings. Our results generalize many known theorems in the literature

Now we begin with some known definitions and facts.

Let (X, d) be a metric space. Then, for $x \in X, A \subset X$, $d(x, A) = \inf\{d(x, y), y \in A\}$. We denote $CB(X)$ as the class of all nonempty bounded subsets of X. Let H be the Hausdorff metric with respect to d, that is, $H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$ for every $A, B \in CB(X)$.

Definition 1.1. *Let (X, d) be a metric space. A subset K of X is called proximinal if, for each $x \in X$, there exists an element $k \in K$ such that $d(x, k) = d(x, K)$.*

The family of all bounded proximinal subsets of X is denoted by $P(X)$.

Definition 1.2. *Let $T : X \to CB(X)$. The map $f : X \to X$ is said to be T-weakly commuting at $x \in X$ if $ffx \in Tfx$.*

Definition 1.3. *Let X be a metric space. A mapping $T : X \to X$ is said to be weakly contractive with respect to $f : X \to X$ if for each $x, y \in X$, $d(Tx, Ty) \leq d(fx, fy) - \phi(d(fx, fy))$, where $\phi : [0.\infty) \to [0.\infty)$ is continuous and non decreasing such that ϕ is positive on $(0, \infty), \phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$.*

Definition 1.4. *A point $x \in X$ is a coincidence point(Common fixed point) of a single valued mapping f and multi-valued mapping T if $f(x) \in T(x)(x = f(x) \in T(x))$.*

Theorem 1.5. *[3]Let (X, d) be a complete metric space and $T : X \to CB(X)$. If α is a function of $(0, \infty)$ to $[0, 1)$ such that $\limsup_{r\to t^+} \alpha(r) < 1$ for every $t \in [0, \infty)$. and if $H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$ for each $x, y \in X$, then T has a fixed point in X.*

Theorem 1.6. *[1]Let (X, d) be a metric space and $T : X \to X$ be a weakly contractive mapping with respect to $f : X \to X$. If the range of f contains the range of T and $f(X)$ is a complete subspace of X,then f and T have coincidence point in X.*

Theorem 1.7. *[3]Let (X, d) be a complete metric space and $T : X \to P(X)$. If α is a monotone increasing function such that $0 \leq \alpha(t) < 1$ for each $t \in (0, \infty)$ and if $H(T(x), T(Y)) \leq \alpha(d(x, y))d(x, y)$ for each $x, y \in X$,thenT has a fixed point in X.*

2 Main results

We begin with the following definition.

Definition 2.1. *Let (X,d) be a metric space, $f : X \to X$ and $T : X \to CB(X)$. T is said to be α- weakly contractive with respect to f if $H(Tx,Ty) \le \alpha(d(fx,fy))d(fx,fy)$ for each $x,y \in X$ and for some $\alpha : (0,\infty) \to [0,1)$ such that $\limsup_{r\to t^+} \alpha(r) < 1$ for every $t \in [0,\infty)$.*

Theorem 2.2. *If $T : X \to CB(X)$ be α- weakly contractive with respect to f, $T(X) = \bigcup_{x\in X} Tx \subset f(X)$ and $f(X)$ is a complete subspace of X, then T and f have a coincidence point in X*

Proof. Let $x_0 \in X$ be arbitrary. Choose $x_1 \in X$ such that $f(x_1) \in T(x_0)$. Continuing the process and having chosen $x_{n+1} \in X$ such that $f(x_{n+1}) \in T(x_n)$, consider

$$\begin{aligned} d(f(x_{n+1}), f(x_{n+2)})) &\le H(T(x_n), T(x_{n+1})) \\ &\le \alpha(d(f(x_n), f(x_{n+1}))d(f(x_n), f(x_{n+1})) \\ &\le d(f(x_n), f(x_{n+1})). \end{aligned}$$

Thus the sequence $\{d_n\} = \{d(f(x_n), f(x_{n-1}))\}$ is a non-increasing sequence of nonnegative real numbers and hence converges to a limit $c \ge 0$. Now, since by our definition(2.1) $\limsup_{r\to t^+} \alpha(r) < 1$, there exists k_0 such that for all $k \ge k_0, \limsup_{t\to c^+} \alpha(d_k) < h < 1$.

$$\begin{aligned} d(f(x_k), f(x_{k+1}) &\le H(T(x_{k-1}), T(x_k)) \\ &\le \alpha(d(f(x_{k-1}), f(x_k))d(f(x_{k-1}), f(x_k)) \\ &= \alpha(d_k)d_k. \end{aligned}$$

That is $d_{k+1} \le \alpha(d_k)d_k$.
Now for $k \ge k_0$,

$$\begin{aligned} d_{k+1} &\le \alpha(d_k)d_k \\ &\le \prod_{i=1}^{k} \alpha(d_i)d_1 \\ &\le h^{k-k_0+1} \prod_{i=1}^{k_0-1} \alpha(d_i)d_1 \\ &= \frac{h^k}{h^{k_0-1}} d_1 \prod_{i=1}^{k_0-1} \alpha(d_i) \\ &\le \frac{d_1}{h^{k_0-1}} h^k \\ &= Ah^k, \end{aligned}$$

where $A = \frac{d_1}{h^{k_0-1}}$ is a generic constant. In the above inequality we make use of the fact that $\alpha(t) < 1$ to omit the product $\prod_{i=1}^{k_0-1} \alpha(d_i)$

Now, consider

$$\begin{aligned} d(f(x_k), f(x_{k+m})) &\leq d(f(x_k), f(x_{k+1})) + d(f(x_{k+1}), f(x_{k+2})) + \\ &\quad .... + d(f(x_{k+m-1}), f(x_{k+m})) \\ &= \sum_{i=k+1}^{k+m} d_i \\ &\leq \sum_{i=k+1}^{k+m} Ah^{i-1} \\ &= A\frac{h^{k+1} - h^{k+m}}{1-h} \\ &= h^k A\frac{h - h^m}{1-h} \\ &\leq h^k, \end{aligned}$$

which tends to zero as $k \to \infty$. Since $f(X)$ is complete, there exists $p \in X$ such that $f(x_k) \to f(p)$. Let $f(p) = q$; then

$$\begin{aligned} d(q, T(p)) &\leq d(q, f(x_k)) + d(f(x_k), T(p)) \\ &\leq d(q, f(x_k)) + H(T(x_{k-1}), T(p)) \\ &\leq d(q, f(x_k)) + \alpha(d(f(x_{k-1}), f(p)))d(f(x_{k-1}), f(p)). \end{aligned}$$

Since both terms in the last expression tend to zero as $k \to \infty$, we obtain $q = f(p) \in T(p)$. □

Remark 2.3. *If $f = Id_x$(the identity map of X in the above Theorem(2.2), then $p \in T(p)$. Thus our Theorem(2.2)is a generalization of the Theorem(1.5).*

Corollary 2.4. *If $T : X \to CB(X)$ be α- weakly contractive with respect to f, where α is a monotone increasing function such that $0 \leq \alpha(t) < 1$ for each $t \in (0, \infty)$, $T(X) = \bigcup_{x \in X} Tx \subset f(X)$ and $f(X)$ is a complete subspace of X, then T and f have a coincidence point in X.*

Remark 2.5. *If we take $f = Id_x$(the identity map of X) in Corollary(2.4), we get the Corollary(2.2) in [3] which is a generalization of Theorem(1.7)*

Theorem 2.6. *Let X be a metric space and T be a α-weakly contractive mapping with respect to f. If f is T-weakly commuting at the coincidence points and $T(X) \subset f(X)$ and $f(X)$ is a complete subspace of X, then f and T have common fixed point in X.*

Proof. By Theorem(2.2), we obtain a point p in X such that $f(p) \in T(p)$. Since f is T-weakly commuting $ff(p) \in Tf(p)$. Let $f(p) = q$. Then $f(q) \in T(q)$. We shall prove that $f(q) = q$. If it is not so, then consider

$$\begin{aligned} d(f(q), q) &\leq H((T(q), T(p)) \\ &\leq \alpha(d(f(q), f(p)))d(f(q), f(p)) \\ &= \alpha(d(f(q), q))d(f(q), q) \\ &< d(f(q), q). \end{aligned}$$

This contradiction leads to the result. Thus $q = f(q) \in T(q)$ is the common fixed point of f and T. □

Theorem 2.7. *Let (X, d) be a metric space and $T : X \mapsto CB(X)$ be a weakly contractive mapping with respect to $f : X \to X$. If the range of f contains the range of T and $f(X)$ is a complete subspace of X, then f and T have coincidence point in X.*

Proof. . Let $x_0 \in X$ be arbitrary. Choose $x_1 \in X$ such that $f(x_1) \in T(x_0)$. This is possible since the range of f contains the range of T. Continuing the process we obtain $x_{n+1} \in X$ such that $f(x_{n+1}) \in T(x_n)$. Now consider

$$\begin{aligned} d(f(x_{n+1}), f(x_{n+2})) &\leq H(T(x_n), T(x_{n+1})) \qquad (1) \\ &\leq d(f(x_n), f(x_{n+1})) - \phi(d(f(x_n), f(x_{n+1})) \\ &\leq d(f(x_n), f(x_{n+1})). \end{aligned}$$

This shows that $d(f(x_n), f(x_{n+1}))$ is a non-increasing sequence of non-negative real numbers and hence tends to a limit $l \geq 0$. If $l > 0$, then we have

$$\begin{aligned} d(f(x_{n+1}), f(x_{n+2})) &\leq d(f(x_n), f(x_{n+1})) - \phi(l) \qquad (2) \\ d(f(x_{n+2}), f(x_{n+3})) &\leq d(f(x_{n+1}), f(x_{n+2})) - \phi(l) \\ &\leq d(f(x_n), f(x_{n+1})) - \phi(l) - \phi(l) \end{aligned}$$

Thus, $d(f(x_{n+N}), f(x_{n+N+1})) \leq d(f(x_n), f(x_{n+1})) - N\phi(l)$,
which is a contradiction for sufficiently large N.
$\therefore \lim_{n\to\infty} d(f(x_n), f(x_{n+1})) = 0$. Further, for $m > n$

$$\begin{aligned} d(f(x_n), f(x_m)) &\leq d(f(x_n), f(x_{n+1})) + d(f(x_{n+1}), f(x_{n+2})) \qquad (3) \\ &\quad + ...d(f(x_{m-1}), f(x_m)). \end{aligned}$$

Now using $\therefore \lim_{n\to\infty} d(f(x_n), f(x_{n+1})) = 0$ and the weak contractivity of T in the above inequality, we get $d(f(x_n), f(x_m)) \to 0$ as $m, n \to 0$.
As $f(X)$ is a complete subspace of X, the sequence $\{f(x_{n+1})\}$ has limit q in $f(X)$.

$\therefore$ there exists $p \in X$ such that $f(p) = q$.
Now,

$$\begin{aligned} d(f(x_{n+1}), T(p)) &\leq H(T(x_n), T(p)) && (4) \\ &\leq d(f(x_n), f(p)) - \phi(d(f(x_n), f(p))). && (5) \end{aligned}$$

Taking limit as $n \to \infty$, we obtain
$d(q, T(p)) \leq d(q, f(p)) - \phi(d(q, f(p)))$.
Since both the terms in the right hand side of the inequality are zero, we have $q \in T(p)$. That is $f(p) \in T(p)$. Thus $p \in X$ is the coincidence point of f and T.

□

Remark 2.8. *If we take T to be a singleton set in Theorem(2.7) we get $p \in X$ such that $f(p) = T(p)$. Thus our Theorem(2.7)is a generalization of corresponding Theorem(1.6) of Ismat Beg et al[1].*

Remark 2.9. *If $f = id_x$(the identity map of X) in Theorem(2.7), then $p = f(p) \in T(p)$ is the fixed point of T.*

Theorem 2.10. *Let X be a metric space and T be a weakly contractive mapping with respect to f. If f is T-weakly commuting at the coincidence points of f and T and $T(X) \subset f(X)$ and $f(X)$ is a complete subspace of X, then f and T have common fixed point in X.*

Proof. By Theorem(2.7), we obtain a point $p \in X$ such that $q = f(p) \in T(p)$. Since f is T commutative, $ff(p) \in Tf(p)$ That is, $f(q) \in T(q)$. Now we show that $f(q) = q$. If it is not so,then consider

$$\begin{aligned} d(f(q), q) &\leq H(T(f(q)), T(p)) && (6) \\ &\leq d(f(q), f(p)) - \phi(d(f(q), f(p))) && (7) \\ &= d(f(q), q)) - \phi(d(f(q), q)) && (8) \\ &< d(f(q), q)), && (9) \end{aligned}$$

which is a contradiction. Thus $q = f(q) \in T(q)$ is the common fixed point of f and T. □

References

[1] Ismat Beg and Mujahid Abbas Coincidence point and invariant approximation for mapping satisfying generalized weak contractive condition,Fixed point Theory and Applications,volume2006(2006)1-7.

[2] Yucheng Liu, Jun Wu and Zhixiang Li,Common fixed points of single valued and multivalued maps,Int.J.Math. Math Sci.2005:19(2005)3045-3055.

[3] Peter Z.Daffer and Hideaki Kaneko,Fixed points of generalized contractive multivalued mappings,J.Math Anal and Math Appl.192,655-666(1995).

[4] Aqeel Ahmad and M.Imdad,Some Common fixed point theorems for mappings and multivalued mappings,J.Math.Anal.Appl,216(1998),546-550.

[5] M.Aamri and D.El Moutawakil,Some new common fixed point theorems under strict contractive conditions,J.Math.Anal.Appl.270(2002) 181-188

[6] G.Jungck, Common fixed point theorems for commuting and compatible maps on Compacta,Proc.Amer.Math.Soc.,103(1988),no.3,977-983.

[7] S.Sessa,On a weak commutativity condition of mappings in fixed point consideration, Instinct Mathematique.publications.Nouville Serie 32(46)(1982),149-153.

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