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Coincidence and Fixed Points of Weakly Contractive Mappings in Fuzzy Metric Spaces

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Abstract

In this paper we define a new weak contractive condition for a pair of self mappings in fuzzy metric space and prove the existence of coincidence and common fixed points.

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1 Introduction

Ever since the notion of fuzzy set was introduced by Zadeh [12] in 1965, the concept of fuzzy metric space was introduced by various authors in different directions. Especially, Deng [1],Erceg [2], Kaleva and Seikkala [5], Karmosil and Michalek [6] have introduced the concept of fuzzy metric space in different ways. George and Veeramani [3] modified the concept of fuzzy metric spaces in the sense of Karmosil and Michalek [6] and defined the Hausdorff topology of fuzzy metric spaces. Consequently they showed every metric induces a fuzzy metric. Grabiec [4] extended the result of Banach contraction principle and Edelstin's fixed point theorem in fuzzy metric spaces in the sense of Karmosil and Michalek [6]. The authors [11] proved the fuzzy analogue fixed point theorems for kannan mappings in the settings of metric spaces. Mishra, Sharma and Singh [9] also proved some fixed point theorems in fuzzy metric spaces. Sushil Sharma [13] proved common fixed point theorems for six mappings. In this paper we define a generalized contractive condition for a self mapping with respect to another and establish the existence of coincidence point. Further we deduce a weak contractive condition for the existence of fixed point for single mapping. We begin with some definitions and preliminary concepts.

Definition 1.1. *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-norm if ([0,1],*) is an abelian toplological monoid with unit* 1 *such that* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2. *A 3-tuple* $(X, M, *)$ *is said to be fuzzy metric space if* X *is an arbitrary set, * is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times [0, \infty)$ *satisfying the following conditions:*
For all $x, y, z \in X$ *and all* $s, t > 0$,

(i) $M(x, y, 0) = 0$,

(ii) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(v) $M(x, y, .) : [0, \infty) \to [0, 1]$ *is left continuous,*

(vi) $\lim_{t\to\infty} M(x, y, t) = 1$.

Definition 1.3. *Let* $(X, M, *)$ *be a fuzzy metric space*
A sequence $\{x_n\}$ *in* X *is called Cauchy sequence if and only if* $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ *for each* $p > 0$,$t > 0$.
A sequence $\{x_n\}$ *in* X *is converging to* x *in* X *if and only if* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *for each* $t > 0$.
A fuzzy metric space $(X, M, *)$ *is said to be complete if and only if every Cauchy sequence in* X *is convergent in* X.

Definition 1.4. *Two mappings* f *and* g *are compatible if and only if* $\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = 1$ *for all* $t > 0$, *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = x_o \in X$.

Definition 1.5. *Two mappings f and g are weakly compatible if they commute at their coincidence points; i.e., if $f(u) = g(u)$ for some $u \in X$,then $fg(u) = gf(u)$.*

Lemma 1.6. *Let f, g be two compatible mappings on X. If $f(x) = g(x)$ for some x in X, then $fg(x) = gf(x)$.*

Definition 1.7. *A point in X is a coincidence point(fixed point)of f and T if $f(x) = T(x)(T(x) = f(x) = x)$.*

2 Main Results

We begin with the following definition.

Definition 2.1. *Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is called weakly contractive with respect to $f : X \to X$ if for each x, y in X, $M(Tx, Ty, t) \geq M(fx, fy, t) + \phi(M(fx, fy, t))$ for all $t > 0$, where $\phi : [0, 1] \to [0, 1]$ is continuous and non-increasing such that $\phi(\alpha) \leq 1 - \alpha$, ϕ is positive in $(0, 1), \phi(0) = 1$ and $\phi(1) = 0$.*

Theorem 2.2. *Let $(X, M, *)$ be a fuzzy metric space and let T be a weakly contractive mapping with respect to f. If the range of f contains the range of T and $f(X)$ is a complete subspace of X, then f and T have a coincidence point in X.*

Proof. Let x_0 be an arbitrary point in X. Choose a point $x_1 \in X$ such that $f(x_1) = T(x_0)$. This is possible since the range of f contains the range of T. Continuing this process and having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_{n+1}) = T(x_n)$. Now consider

$$\begin{aligned} M(f(x_{n+1}), f(x_{n+2}), t) &= M(T(x_n), T(x_{n+1}), t) \\ &\geq M(f(x_n), f(x_{n+1}), t) + \phi(M(f(x_n), f(x_{n+1}), t)) \\ &\geq M(f(x_n), f(x_{n+1}), t). \end{aligned} \tag{1}$$

This implies that $\{M(f(x_{n+1}), f(x_{n+2}), t)\}$ is a nondecreasing sequence of positive real numbers with lower bound 0 and upper bound 1. Hence it converges to a limit $0 < l \leq 1$. If $l \neq 1$, then we have

$$\begin{aligned} M(f(x_{n+1}), f(x_{n+2}), t) &\geq M(f(x_n), f(x_{n+1}), t) + \phi(l) \\ M(f(x_{n+2}), f(x_{n+3}), t) &\geq M(f(x_{n+1}), f(x_{n+2}), t) + \phi(l) \\ &\geq M(f(x_n), f(x_{n+1}), t) + 2\phi(l) \end{aligned}$$

Thus, $M(f(x_{n+N}), f(x_{n+N+1}), t) \geq M(f(x_n), f(x_{n+1}), t) + N\phi(l)$,....................(2)
which is a contradiction for sufficiently large N.
Therefore, $\lim_{n\to\infty} M(f(x_n), f(x_{n+1}), t) = 1$ for all $t > 0$..............................(3)

Further,from (2)we have
$M(f(x_{n+N}), f(x_{n+N+1}), t) \geq M(f(x_n), f(x_{n+1}), t)$ ....(4)
Now using (4) and the property (iv) in Definition(1.2),
we obtain

$$\begin{aligned} M(f(x_{n+p}), f(x_n), t) &\geq M(f(x_{n+1}), f(x_n), \frac{t}{2}) * M(f(x_{n+1}), f(x_n), \frac{t}{2^2}) \\ & \quad *\ldots\ldots\ldots\ldots M(f(x_{n+1}), f(x_n), \frac{t}{2^{p-1}}) \end{aligned}$$

where $t > 0$.
Taking limit as $n \to \infty$ and using (3), we get

$\lim_{n\to\infty} M(f(x_{n+p}), f(x_n), t) = 1$. Thus $\{f(x_n)\}$ is a Cauchy sequence. Since $f(X)$ is complete, there exists $p \in X$ such that $\lim_{n\to\infty} f(x_n) = f(p)$. Let $f(p) = q$. Now,

$$\begin{aligned} M(f(x_{n+1}), T(p), t) &= M(T(x_n), T(p), t) \\ &\geq M(f(x_n), f(p), t) + \phi(M(f(x_n), f(p), t)) \end{aligned}$$

Taking limit as $n \to \infty$, we obtain
$M(q, T(p), t) \geq 1 + \phi(1)$. That is $M(q, T(p), t) = 1$.
Therefore, $q = T(p) = f(p)$. □

Theorem 2.3. *Let $(X, M, *)$ be a fuzzy complete metric space and let $T : X \to X$. If $M(Tx, Ty, t) \geq M(x, y, t) + \phi(M(x, y, t))$ for each $x, y \in X$ and for all $t > 0$, where $\phi : [0, 1] \to [0, 1]$ is continuous and nonincreasing such that $\phi(\alpha) \leq 1 - \alpha$, ϕ is positive in $(0, 1), \phi(0) = 1$ and $\phi(1) = 0$,then T has a fixed point in X.*

Proof. If $f = id_x$(the identity map of X)in Theorem(2.2),then $p = T(p)$ is the fixed point of T. □

Theorem 2.4. *Let $(X, M, *)$ be a fuzzy metric space and let T be a weakly contractive mapping with respect to f. If T and f are weakly compatible and $T(X) \subset f(X)$ and $f(X)$ is a complete subspace of X, then T and f have a common fixed point in X.*

Proof. By Theorem(2.2), we get a point $p \in X$ such that $f(p) = T(p) = q$(say). Then by weak compatibility $Tf(p) = fT(p)$. Further $ff(p) = fT(p)$.This implies $ff(p) = Tf(p)$. Therefore $f(q) = T(q)$. Now we show that $f(q) = q$. If it is not so, consider

$$\begin{aligned} M(f(q), q, t) &= M(T(q), T(p), t) \\ &\geq M(f(q), f(p), t) + \phi(M(f(q), f(p), t)) \\ &\geq M(f(q), q, t) + \phi(M(f(q), q, t)) \\ &> M(f(q), q, t). \end{aligned}$$

This contradiction leads to the result. □

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