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Coincidence and Fixed Points of Weakly Contractive Mappings in Intutionistic Fuzzy Metric Spaces

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Abstract

In this paper we define a new weak contractive condition for a pair of self mappings in Intutionistic fuzzy metric space and prove the existence of coincidence and common fixed points

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1 Introduction

Ever since the notion of fuzzy set was introduced by Zadeh [7] in 1965, the concept of fuzzy metric space was introduced by various authors in different directions. Especially, Deng [1],Erceg [2], Kaleva and Seikkala [5], Karmosil and Michalek [6] have introduced the concept of fuzzy metric space in different ways. George and Veeramani [3] modified the concept of fuzzy metric spaces in the sense of Karmosil and Michalek [6] and defined the Hausdorff topology of fuzzy metric spaces. In 1983 Atanassov [13] defined intutionistic fuzzy set. Subsequently Park [11] generalized the concept of fuzzy metric space by introducing the intutionistic fuzzy metric space. Alaca et al[12] and Park introduced the notion of cauchy sequences and proved the well known fixed point theorems of Banach in the context of intutionistic fuzzy metric space. In this paper we introduce weak contractivity condition for one mapping with respect to another in Intutionistic fuzzy metric space and prove the existence of coincidence and common fixed points in the intutionistic fuzzy metric space. We begin with some definitions and preliminary concepts.

Definition 1.1. *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-norm if ([0,1],*) is an abelian toplological monoid with unit* 1 *such that* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2. *A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is called continuous* $t - conorm$ *if it satisfies the following conditions:*

(a) $\diamond$ *is commutative and associative;*

(b) $\diamond$ *is continuous;*

(c) $a \diamond 0 = a$ *for all* $a \in [0,1]$*;*

(d) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

Example of $t - conorm$ is $a \diamond b = \min\{1, a + b\}$ where $a, b \in [0,1]$.

Definition 1.3. *A 5-tuple* $(X, M, N, *, \diamond)$ *is said to be an intutionistic fuzzy metric space if* X *is an arbitrary set, * is a continuous t-norm,* $\diamond$ *is a continuous* $t - conorm$ *and* M, N *are a fuzzy sets on* $X^2 \times [0, \infty)$ *satisfying the following conditions: for all* $x, y, z \in X$ *and all* $s, t > 0$,

(1) $M(x, y, t) + N(x, y, t) \leq 1$*;*

(2) $M(x, y, 0) = 0$*;*

(3) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$*;*

(4) $M(x, y, t) = M(y, x, t)$*;*

(5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$*;*

(6) $M(x, y, .) : [0, \infty) \to [0,1]$ *is left continuous;*

(7) $\lim_{t\to\infty} M(x, y, t) = 1$*;*

(8) $N(x, y, 0) = 1$*;*

(9) $N(x,y,t) = 0$ *for all* $t > 0$ *if and only if* $x = y$;

(10) $N(x,y,t) = N(y,x,t)$;

(11) $N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$;

(12) $N(x,y,.) : [0,\infty) \to [0,1]$ *is right continuous;*

(13) $\lim_{t\to\infty} N(x,y,t) = 0$.

(M,N) *is called an intutionistic fuzzy metric on* X. *The functions* $M(x,y,t)$ *and* $N(x,y,t)$ *denote the degree of nearness and non-nearness between* x *and* y *with respect to* t *respectively.*

Definition 1.4. *Let* $(X,M,N,*,\diamond)$ *be a intutionistic fuzzy metric space*
A sequence $\{x_n\}$ *in* X *is called Cauchy sequence if and only if* $\lim_{n\to\infty} M(x_{n+p},x_n,t) = 1$ *and* $\lim_{n\to\infty} N(x_{n+p},x_n,t) = 0$ *for each* $p > 0$, $t > 0$.
A sequence $\{x_n\}$ *in* X *is converging to* x *in* X *if and only if* $\lim_{n\to\infty} M(x_n,x,t) = 1$ *and* $\lim_{n\to\infty} N(x_n,x,t) = 0$ *for each* $t > 0$.
An intutionistic fuzzy metric space $(X,M,N,*,\diamond)$ *is said to be complete if and only if every Cauchy sequence in* X *is convergent in* X.

Definition 1.5. *Two mappings* f *and* g *are compatible if and only if* $\lim_{n\to\infty} M(fg(x_n),gf(x_n),t) = 1$ *and* $\lim_{n\to\infty} N(fg(x_n),gf(x_n),t) = 0$ *for all* $t > 0$, *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = x_o \in X$.

Definition 1.6. *Two mappings* f *and* g *are compatible if* $f(x_0) = g(x_0)$ *for some* $x_0 \in X$, *then* $fg(x_0) = gf(x_0)$

Lemma 1.7. *Let* f,g *be two compatible mappings on* X. *If* $f(x) = g(x)$ *for some* x *in* X, *then* $fg(x) = gf(x)$.

Definition 1.8. *A point in* X *is a coincidence point(fixed point)of* f *and* T *if* $f(x) = T(x)(T(x) = f(x) = x)$.

2 Main Results

We begin with the following definition.

Definition 2.1. *Let* $(X,M,N,*,\diamond)$ *be an intutionistic fuzzy metric space. A mapping* $T : X \to X$ *is called weakly contractive with respect to* $f : X \to X$ *if for each* x,y *in* X, $M(Tx,Ty,t) \geq M(fx,fy,t) + \phi(M(fx,fy,t))$ *and* $N(Tx,Ty,t) \leq N(fx,fy,t) - \psi(N(fx,fy,t))$ *for all* $t > 0$, *where* $\phi : [0,1] \to$

$[0,1]$ is continuous and nonincreasing such that $\phi(\alpha) \leq 1-\alpha$, ϕ is positive in $(0,1), \phi(0) = 1$ and $\phi(1) = 0$ and $\psi : [0,1] \to [0,1]$ is continuous and nondecreasing such that $\psi(\alpha) \leq \alpha$,ψ is positive in $(0,1), \psi(0) = 0$ and $\psi(1) =$ 1.

Theorem 2.2. *Let $(X, M, N, *, \diamond)$ be an intutionistic fuzzy metric space and let $T : X \to X$ be a weakly contractive mapping with respect to $f : X \to X$. If the range of f contains the range of T and $f(X)$ is a complete subspace of X, then f and T have a coincidence point in X.*

Proof. Let x_0 be an arbitrary point in X. Choose a point $x_1 \in X$ such that $f(x_1) = T(x_0)$. This is possible since the range of f contains the range of T. Continuing this process and having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_{n+1}) = T(x_n)$. Now consider

$$\begin{aligned} M(f(x_{n+1}), f(x_{n+2}), t) &= M(T(x_n), T(x_{n+1}), t) \\ &\geq M(f(x_n), f(x_{n+1}), t) + \phi(M(f(x_n), f(x_{n+1}), t)) \\ &\geq M(f(x_n), f(x_{n+1}), t). \qquad (1) \end{aligned}$$

This implies that $\{M(f(x_{n+1}), f(x_{n+2}), t)\}$ is a nondecreasing sequence of positive real numbers with lower bound 0 and upper bound 1. Hence it converges to a limit $0 < l \leq 1$. If $l \neq 1$, then we have

$$\begin{aligned} M(f(x_{n+1}), f(x_{n+2}), t) &\geq M(f(x_n), f(x_{n+1}), t) + \phi(l) \\ M(f(x_{n+2}), f(x_{n+3}), t) &\geq M(f(x_{n+1}), f(x_{n+2}), t) + \phi(l) \\ &\geq M(f(x_n), f(x_{n+1}), t) + 2\phi(l) \end{aligned}$$

Thus, $M(f(x_{n+N}), f(x_{n+N+1}), t) \geq M(f(x_n), f(x_{n+1}), t) + N\phi(l)$,....................(2)
which is a contradiction for sufficiently large N.
Therefore, $\lim_{n\to\infty} M(f(x_n), f(x_{n+1}), t) = 1$ for all $t > 0$..............................(3)

Further,from (2)we have
$M(f(x_{n+N}), f(x_{n+N+1}), t) \geq M(f(x_n), f(x_{n+1}), t)$ ....(4)
Now using (4) and the property (iv) in Definition(1.3),
we obtain

$$\begin{aligned} M(f(x_{n+p}), f(x_n), t) \geq\ & M(f(x_{n+1}), f(x_n), \frac{t}{2}) * M(f(x_{n+1}), f(x_n), \frac{t}{2^2}) \\ & *...........M(f(x_{n+1}), f(x_n), \frac{t}{2^{p-1}}) \end{aligned}$$

where $t > 0$.
Taking limit as $n \to \infty$ and using (3), we get

$\lim_{n\to\infty} M(f(x_{n+p}), f(x_n), t) = 1.$

Thus $\{f(x_n)\}$ is a Cauchy sequence. Since $f(X)$ is complete, there exists $p \in X$ such that $\lim_{n\to\infty} f(x_n) = f(p)$. Let $f(p) = q$. Now,

$$\begin{aligned} M(f(x_{n+1}), T(p), t) &= M(T(x_n), T(p), t) \\ &\geq M(f(x_n), f(p), t) + \phi(M(f(x_n), f(p), t)) \end{aligned}$$

Taking limit as $n \to \infty$, we obtain
$M(q, T(p), t) \geq 1 + \phi(1)$. That is $M(q, T(p), t) = 1$.
Now consider

$$\begin{aligned} N(f(x_{n+1}), f(x_{n+2}), t) &= N(T(x_n), T(x_{n+1}), t) \\ &\leq N(f(x_n), f(x_{n+1}), t) - \psi(N(f(x_n), f(x_{n+1}), t)) \\ &\leq N(f(x_n), f(x_{n+1}), t). \end{aligned} \tag{2}$$

This implies that $\{N(f(x_{n+1}), f(x_{n+2}), t)\}$ is a nonincreasing sequence of positive real numbers with lower bound 0. Hence it converges to a limit $c \geq o$. We show that $c = 0$. If it is not so, then

$$\begin{aligned} N(f(x_{n+1}), f(x_{n+2}), t) &\leq N(f(x_n), f(x_{n+1}), t) - \psi(c) \\ N(f(x_{n+2}), f(x_{n+3}), t) &\leq N(f(x_{n+1}), f(x_{n+2}), t) - \psi(c) \\ &\leq N(f(x_n), f(x_{n+1}), t) - 2\psi(c) \end{aligned}$$

Thus $N(f(x_{n+K}), f(x_{n+K+1}), t) \leq N(f(x_n), f(x_{n+1}), t) - K\psi(c)$,....................(2)
which is a contradiction for sufficiently large K.

$$\begin{aligned} N(f(x_{n+p}), f(x_n), t) \leq\ & N(f(x_{n+1}), f(x_n), \frac{t}{2}) \diamond N(f(x_{n+1}), f(x_n), \frac{t}{2^2}) \\ & \diamond ....N(f(x_{n+1}), f(x_n), \frac{t}{2^{p-1}}) \end{aligned}$$

where $t > 0$.
Taking limit as $n \to \infty$ and using (3), we get
$\lim_{n\to\infty} N(f(x_{n+p}), f(x_n), t) = 0$.

Thus $\{f(x_n)\}$ is a Cauchy sequence. Since $f(X)$ is complete, there exists $p \in X$ such that $\lim_{n\to\infty} f(x_n) = f(p)$. Let $f(p) = q$. Now,

$$\begin{aligned} N(f(x_{n+1}), T(p), t) &= N(T(x_n), T(p), t) \\ &\leq N(f(x_n), f(p), t) - \psi(N(f(x_n), f(p), t)) \end{aligned}$$

Taking limit as $n \to \infty$, we obtain
$N(q, T(p), t) \leq 0 + \psi(0)$. That is $N(q, T(p), t) = 0$.
Therefore, $q = T(p) = f(p)$. $\square$

Theorem 2.3. *Let* $(X, M, *, \diamond)$ *be an intutionistic fuzzy metric space which is complete and let* $T : X \to X$. *If* $M(Tx, Ty, t) \geq M(x, y, t) + \phi(M(x, y, t))$ *and* $N(Tx, Ty, t) \leq N(x, y, t) - \psi(N(x, y, t))$ *for each* $x, y \in X$ *and for all* $t > 0$, *where*

(i) $\phi : [0, 1] \to [0, 1]$ *is continuous and nonincreasing such that* $\phi(\alpha) \leq 1-\alpha$, ϕ *is positive in* $(0, 1), \phi(0) = 1$ *and* $\phi(1) = 0$

(ii) $\psi : [0, 1] \to [0, 1]$ *is continuous and nondecreasing such that,* $\psi(\alpha) \geq \alpha$ ψ *is positive in* $(0, 1), \psi(0) = 0$ *and* $\psi(1) = 1$,

then T has a fixed point in X.

Proof. If $f = id_x$(the identity map of X)in Theorem(2.2),then $p = T(p)$ is the fixed point of T. □

Theorem 2.4. *Let* $(X, M, *, \diamond)$ *be an intutionistic fuzzy metric space and let* T *be a weakly contractive mapping with respect to* f. *If* T *and* f *are weakly compatible and* $T(X) \subset f(X)$ *and* $f(X)$ *is complete subspace of* X, *then* T *and* f *have a common fixed point in* X.

Proof. By Theorem(2.2), we get a point $p \in X$ such that $f(p) = T(p) = q$(say). Then by Lemma(1.7),$Tf(p) = fT(p)$. Further $ff(p) = fT(p)$.This implies $ff(p) = Tf(p)$. Therefore $f(q) = T(q)$. Now we show that $f(q) = q$. If it is not so, consider

$$\begin{aligned} M(f(q), q, t) &= M(T(q), T(p), t) \\ &\geq M(f(q), f(p), t) + \phi(M(f(q), f(p), t)) \\ &\geq M(f(q), q, t) + \phi(M(f(q), q, t)) \\ &> M(f(q), q, t). \end{aligned}$$

$$\begin{aligned} N(f(q), q, t) &= N(T(q), T(p), t) \\ &\leq N(f(q), f(p), t) - \psi(N(f(q), f(p), t)) \\ &\leq N(f(q), q, t) - \psi(N(f(q), q, t)) \\ &< N(f(q), q, t). \end{aligned}$$

This contradiction leads to the result. □

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