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Isometry of a Sequence Space Generated by a Difference Operator

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Abstract

The difference sequence spaces $\ell_\infty(\triangle), c(\triangle)$, and $c_0(\triangle)$ were studied by H. Kizmaz [3]. These difference sequence spaces were then generalized by Colak and Et [2]. In this study, we consider the difference sequence space $\ell_p(\triangle^m)$. Using a recursive method, we will show that if $\ell_p(\triangle^m)$ is equipped with an appropriate norm then this sequence space is linearly isometric to the usual sequence space ℓ_p.

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1 Introduction

All throughout this paper we let $p \in [1, \infty)$. By ω, we shall denote the space of all sequences $x = (x_k)$, where $x_k \in \mathbb{C}$ for all $k \in \mathbb{N}$. Given $x \in \omega$, define

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$$

and let

$$\ell_p = \{x = (x_k) : \|x\|_p < \infty\}.$$

We define the linear difference operator $\triangle : \omega \to \omega$ which maps a sequence $x \in \omega$ into a sequence $\triangle x = (\triangle x_k) \in \omega$ having components

$$\triangle x_k = x_k - x_{k+1}.$$

For $m \geq 2$, the linear operator $\triangle^m : \omega \to \omega$ is defined recursively as the composition $\triangle^m = \triangle \circ \triangle^{m-1}$. One can easily check that for $m \geq 1$ and $x \in \omega$ we have the following Binomial representation

$$\triangle^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v},$$

for all $k \in \mathbb{N}$.

Given $m \in \mathbb{N}$ we define the sequence space

$$\ell_p(\triangle^m) = \{x = (x_k) : \triangle^m x \in \ell_p\}$$

and for $x \in \ell_p(\triangle^m)$ we let

$$\|x\|_{p,\triangle^m} = \left(\sum_{i=1}^{m} |x_i|^p + \|\triangle^m x\|_p^p \right)^{1/p}. \tag{1}$$

It can be easily seen that the pair $(\ell_p(\triangle^m), \|\cdot\|_{p,\triangle^m})$ is a normed space.

If ℓ_∞, c, and c_0 are the linear spaces of bounded, convergent, and null sequences $x = (x_k)$ having complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, then we similarly define the difference sequence spaces

$$\begin{aligned} \ell_\infty(\triangle^m) &= \{x = (x_k) : \triangle^m x \in \ell_\infty\}, \\ c(\triangle^m) &= \{x = (x_k) : \triangle^m x \in c\}, \\ c_0(\triangle^m) &= \{x = (x_k) : \triangle^m x \in c_0\}. \end{aligned}$$

We have the following inclusions $\ell_p(\triangle^m) \subset \ell_\infty(\triangle^m)$ and $\ell_p(\triangle^m) \subset c_0(\triangle^m) \subset c(\triangle^m)$. The difference sequence spaces $\ell_\infty(\triangle^m)$, $c(\triangle^m)$, and $c_0(\triangle^m)$ have been considered by Colak and Et [2] and they showed that these are Banach spaces with norm

$$\|x\|_\triangle = \sum_{i=1}^{m} |x_i| + \|\triangle^m x\|_\infty.$$

For Euler difference sequence spaces and sequence spaces generated by a sequence of Orlicz functions, the reader may consult Qamaruddin and Saifi [4] and Altay and Polat [1], respectively. In this paper, we will show that $\ell_p(\triangle^m)$ with norm $\|\cdot\|_{p,\triangle^m}$ is a Banach space linearly isometric to the ordinary sequence space ℓ_p. Furthermore, a sufficient condition for the inclusion $\ell_p(\triangle^m) \subset \ell_p(\mathcal{M}, \triangle^m)$, where $\mathcal{M}$ is a family of Orlicz functions satisfying the Δ_2-condition, shall be given.

2 Main Results

Theorem 2.1. *The sequence space $\ell_p(\triangle^m)$ equipped with the norm $\|\cdot\|_{p,\triangle^m}$ is a Banach space.*

Proof. Let $(x^{(n)}) = ((x_k^{(n)}))$ be a Cauchy sequence in $\ell_p(\triangle^m)$. Then given $\epsilon > 0$ we can find a positive integer N such that $\|x^{(n)} - x^{(r)}\|_{p,\triangle^m} < \epsilon$ whenever $n, r \geq N$, that is,

$$\left(\sum_{i=1}^{m} |x_i^{(n)} - x_i^{(r)}|^p + \|\triangle^m x^{(n)} - \triangle^m x^{(r)}\|_p^p \right)^{1/p} < \epsilon,$$

for $n, r \geq N$. Since

$$|x_i^{(n)} - x_i^{(r)}| \leq \|x^{(n)} - x^{(r)}\|_{p,\triangle^m}$$

for all $i = 1, 2, \dots, m$ and

$$\|\triangle^m x^{(n)} - \triangle^m x^{(r)}\|_p \leq \|x^{(n)} - x^{(r)}\|_{p,\triangle^m},$$

it follows that $(x_i^{(n)})$ and $(\triangle^m x^{(n)})$ are Cauchy sequence in $\mathbb{C}$ and ℓ_p, respectively. The completeness of the spaces $\mathbb{C}$ and ℓ_p imply the existence of elements $y_i \in \mathbb{C}$, $i = 1, 2, \dots, m$, and $z = (z_k) \in \ell_p$ such that

$$\lim_{n\to\infty} |x_i^{(n)} - y_i| = 0 \tag{2}$$

for all $i = 1, 2, \dots, m$ and

$$\lim_{n\to\infty} \|\triangle^m x^{(n)} - z\|_p = 0. \tag{3}$$

Furthermore, since $|\triangle^m x_k^{(n)} - z_k| \leq \|\triangle^m x^{(n)} - z\|_p$ it follows from Equation (3) that $|\triangle^m x_k^{(n)} - z_k| \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$.

Next, we will find a recursive formula for the limit of $x_{m+i}^{(n)}$, $i \geq 1$, as $n \to \infty$. Notice that

$$(-1)^m x_{m+1}^{(n)} = \triangle^m x_1^{(n)} - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} x_{v+1}^{(n)}$$

and so

$$w_{m+1} := \lim_{n\to\infty} x_{m+1}^{(n)} = (-1)^m \left[z_1 - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} y_{v+1} \right].$$

Suppose that $w_{m+1}, \dots, w_{m+k-1}$, $1 < k \leq m$, have been constructed where

$$w_{m+i} := \lim_{n\to\infty} x^{(n)}_{m+i}, \quad i = 1, 2, \dots, k-1.$$

Using these we have, for $1 < k \leq m$,

$$\begin{aligned} w_{m+k} \;:=\; \lim_{n\to\infty} x^{(n)}_{m+k} \;=\; & (-1)^m \Bigg[z_k - \sum_{v=0}^{m-k} (-1)^v \binom{m}{v} y_{v+k} \\ & - \sum_{v=1}^{k-1} (-1)^{m-k+v} \binom{m}{m-k+v} w_{m+v} \Bigg]. \end{aligned}$$

On the other hand, for $k > m$ we have

$$(-1)x^{(n)}_{m+k} = \triangle^m x^{(n)}_k - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} x^{(n)}_{v+k}$$

so that

$$w_{m+k} = \lim_{n\to\infty} x^{(n)}_{m+k} = (-1)^m \left[z_k - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} w_{k+v} \right].$$

Let $w = (y_1, \dots, y_m, w_{m+1}, w_{m+2}, \dots)$. We claim that $w \in \ell_p(\triangle^m)$, that is, $\triangle^m w \in \ell_p$. First, observe that

$$\begin{aligned} (\triangle^m w)_1 &= \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} y_{v+1} + (-1)^m w_{m+1} \\ &= \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} y_{v+1} + \left[z_1 - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} y_{v+1} \right] \\ &= z_1. \end{aligned}$$

Also, for $k = 2, \dots, m$, we have

$$\begin{aligned} (\triangle^m w)_k &= \sum_{v=0}^{m-k} (-1)^v \binom{m}{v} y_{v+k} + \sum_{v=m-k+1}^{m-1} (-1)^v \binom{m}{v} w_{v+k} + (-1)^m w_{m+k} \\ &= z_k. \end{aligned}$$

Similarly, for $k > m$ we obtain

$$\begin{aligned} (\triangle^m w)_k &= \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} w_{v+k} + (-1)^m w_{m+k} \\ &= z_k. \end{aligned}$$

Therefore we have shown that $\triangle^m w = z \in \ell_p$. Finally, it remains to show that $\|x^{(n)} - w\|_{p,\triangle^m} \to 0$ as $n \to \infty$. This follows directly from Equations (2) and (3) and the following computations

$$\begin{aligned}\lim_{n\to\infty} \|x^{(n)} - w\|^p_{p,\triangle^m} &= \lim_{n\to\infty} \left(\sum_{k=1}^{m} |x_k^{(n)} - y_k|^p + \|\triangle^m x^{(n)} - \triangle^m w\|_p^p \right) \\ &= \sum_{k=1}^{m} \lim_{n\to\infty} |x_k^{(n)} - y_k|^p + \lim_{n\to\infty} \|\triangle^m x^{(n)} - z\|_p^p \\ &= 0.\end{aligned}$$

This completes the proof of the theorem. □

Theorem 2.2. *The sequence spaces* $(\ell_p(\triangle^m), \|\cdot\|_{p,\triangle^m})$ *and* $(\ell_p, \|\cdot\|_p)$ *are linearly isometric.*

Proof. Consider the map $T : \ell_p(\triangle^m) \to \ell_p$ defined by $Ty = x$, where $y = (y_k) \in \ell_p(\triangle^m)$ and $x = (x_k)$ with

$$x_k = \begin{cases} y_k, & \text{if } 1 \le k \le m; \\ \triangle^m y_{k-m}, & \text{if } k > m. \end{cases}$$

The linearity of the difference operator $\triangle$ implies the linearity of T. If $y \in \ell_p(\triangle^m)$ and $Ty = x$ then

$$\begin{aligned}\|Ty\|_p^p = \|x\|_p^p &= \sum_{k=1}^{m} |y_k|^p + \sum_{k=m+1}^{\infty} |\triangle^m y_{k-m}|^p \\ &= \sum_{k=1}^{m} |y_k|^p + \sum_{k=1}^{\infty} |\triangle^m y_k|^p \\ &= \|y\|^p_{p,\triangle^m} < \infty.\end{aligned}$$

This shows that T is well-defined and it is also norm preserving. Now, we are going to show that T is one-to-one and onto. Assume that $Ty = 0$. Then it follows that

$$\triangle^m y_k = 0 \quad \text{for all} \quad k \ge 1, \tag{4}$$
$$y_1 = y_2 = \cdots = y_m = 0. \tag{5}$$

We note that the difference equation (4) with initial conditions (5) implies that $y_k = 0$ for all $k \ge 1$, that is, $y = (0, 0, \ldots)$. Hence T is one-to-one.

Suppose that $x = (x_k) \in \ell_p$. Define the sequence $y = (y_k)$ as follows. For $k = 1, 2, \ldots, m$, let $y_k = x_k$. The succeeding terms of the sequence y is then defined recursively by

$$\begin{aligned} y_{m+1} &= (-1)^m \left[x_{m+1} - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} x_{v+k} \right] \\ y_{m+k} &= (-1)^m \Bigg[x_{m+k} - \sum_{v=0}^{m-k} (-1)^v \binom{m}{v} x_{v+k} \\ &\quad - \sum_{v=1}^{k-1} (-1)^{m-k+v} \binom{m}{m-k+v} y_{m+v} \Bigg], \quad 1 < k \le m, \end{aligned}$$

and

$$y_{m+k} = (-1)^m \left[x_{m+k} - \sum_{v=0}^{m-1} (-1)^v \binom{m}{v} y_{v+k} \right], \quad k > m.$$

Using a similar argument as in the proof of the previous theorem, we can show that

$$\triangle^m y_k = x_{k+m}$$

for $k \in \mathbb{N}$. Hence it follows that $Ty = x$. Furthermore,

$$\begin{aligned} \|\triangle^m y\|_p^p &= \sum_{k=1}^{\infty} |\triangle^m y_k|^p \\ &= \sum_{k=1}^{\infty} |x_{k+m}|^p \\ &\le \|x\|_p^p < \infty, \end{aligned}$$

so that $y \in \ell_p(\triangle^m)$. Hence, T is onto. Therefore $\ell_p(\triangle^m)$ and ℓ_p are linearly isometric. □

An **Orlicz function** $M : [0, \infty) \to [0, \infty)$ is a continuous, convex, non-decreasing function with $M(u) = 0$ if and only if $u = 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function is said to satisfy the **Δ_2-condition** if there exists a positive constant K such that $M(2u) \le KM(u)$ for all $u \ge 0$. Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions satisfying the Δ_2-condition. Define the sequence spaces

$$\ell_p(\mathcal{M}) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |M_k(|x_k|/\rho)|^p < \infty \right\}$$

and

$$\ell_p(\mathcal{M}, \triangle^m) = \{ x = (x_k) : \triangle^m x \in \ell_p(\mathcal{M}) \}.$$

Theorem 2.3. *Assume that $\mathcal{M} = (M_k)$ is a sequence of Orlicz functions satisfying the Δ_2-condition. If*

$$\sum_{k=1}^{\infty} |M_k(t/\rho)|^p < \infty \tag{6}$$

for all $t, \rho > 0$ then $\ell_p(\triangle^m) \subset \ell_p(\mathcal{M}, \triangle^m)$.

Proof. Suppose that condition (6) holds and let $x = (x_k) \in \ell_p(\triangle^m)$. Then it follows that

$$\sum_{k=1}^{\infty} |\triangle^m x_k|^p < \infty.$$

The convergence of the above series implies that

$$\lim_{k\to\infty} |\triangle^m x_k| = 0.$$

Then we can find a positive integer N such that $|\triangle^m x_k| \leq 1$ for all $k \geq N$. Let

$$M = \max(|\triangle^m x_1|, \ldots , |\triangle^m x_{N-1}|, 1).$$

Then $|\triangle^m x_k| \leq M$ for all $k \in \mathbb{N}$. For $\rho > 0$, using the monotonicity of M_k, we have $M_k(|\triangle^m x_k|/\rho) \leq M_k(M/\rho)$ for all $k \in \mathbb{N}$. This inequality implies that

$$\sum_{k=1}^{\infty} |M_k(|\triangle^m x_k|/\rho)|^p \leq \sum_{k=1}^{\infty} |M_k(M/\rho)|^p$$

and from equation (6) this estimate implies that $\triangle^m x \in \ell_p(\mathcal{M})$, that is, $x \in \ell_p(\mathcal{M}, \triangle^m)$. Therefore the inclusion $\ell_p(\triangle^m) \subset \ell_p(\mathcal{M}, \triangle^m)$ holds. □

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