International Mathematical Forum, 5, 2010, no. 43, 2131 - 2136

Positive Solutions for Boundary Value Problems of Higher Order Differential Equations

S. N. Odda

Department of Mathematics, Faculty of Computer Science
Qassim University, Burieda, Saudi Arabia
nabhan100@yahoo.com

Abstract

In this paper, we investigate the problem of existence and nonexistence of positive solutions for the nonlinear boundary value problem:

$$u^{(n)}(t)+\lambda a(t)f(u(t))=0, \qquad 0<t<1,$$

$$u(0)=u'(0)=u''(0)=u'''(0)=\cdots\cdots=u^{(n-2)}(0)=0, \quad u'''(1)=0.$$

Our analysis relies on Krasnoselskii's fixed theorem of cone. An example is also given to illustrate the main results.

Mathematics Subject Classification: 34B15, 34B18

Keywords: Higher-order differential equations, Boundary value problems, Krasnoselskii's fixed point theorem

1. Introduction

The purpose of this paper is to establish the existence of positive solutions to nonlinear higher-order boundary value problems:

$$u^{(n)}(t)+\lambda a(t)f(u(t))=0, \qquad 0<t<1, \tag{1}$$

$$u(0)=u'(0)=u''(0)=u'''(0)=\cdots\cdots=u^{(n-2)}(0)=0, \quad u'''(1)=0, \tag{2}$$

Where λ is a positive parameter. Throughout the paper, we assume that
$C1$: $f:[0,\infty)\to[0,\infty)$ is continuous

$C2:$ a : $(0,1)\to[0,\infty)$ is continuous function such that $\int_0^1 a(t)dt > 0$.

We present here some notations and lemmas that will be used in the proof our main results. These definitions and lemmas can be found in the recent literature [2,3,5].

Definition 1. Let *E* be a real Banach space. A nonempty closed set $K\subset E$ is called a *cone* of *E* if it satisfies the following conditions:

(1) $x\in K,\ \sigma\ge 0$ implies $\sigma x\in K$;

(2) $x\in K, -x\in K$ implies $x=0$.

Definition 2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets**.**

Lemma1. Let *E* be a Banach space and $K\subset E$ is a cone in *E*. Assume that Ω_1 and Ω_2 are open subsets of *E* with $0\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$. Let $T: K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$ be a completely continuous operator. In addition suppose either:

(H1) $\|Tu\|\le\|u\|, \forall u\in K\cap\partial\Omega_1$ and $\|Tu\|\ge\|u\|, \forall u\in K\cap\partial\Omega_2$ or

(H2) $\|Tu\|\le\|u\|, \forall u\in K\cap\partial\Omega_2$ and $\|Tu\|\ge\|u\|, \forall u\in K\cap\partial\Omega_1$

holds. Then *T* has a fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$.

2. Green functions and their properties

Lemma 2. Let $y\in C[0,1]$ then the boundary value problem

$$u^{(n)}(t)+y(t)=0,\qquad 0<\mathrm{t}<1, \tag{3}$$

$$u(0)=u'(0)=u''(0)=u'''(0)=\cdots\cdots=u^{(n-2)}(0)=0,\quad \mathrm{u}'''(1)=0, \tag{4}$$

has a unique solution

$$u(t)=\int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s)=\begin{cases}\dfrac{t^{(n-1)}(1-s)^{(n-4)}}{(n-1)!} - \dfrac{(t-s)^{(n-1)}}{(n-1)!} & \text{if } 0\le \mathrm{s}\le \mathrm{t}\le 1;\\[2mm] \dfrac{t^{(n-1)}(1-s)^{(n-4)}}{(n-1)!} & \text{if } 0\le \mathrm{t}\le \mathrm{s}\le 1;\end{cases}$$

Proof: Applying the Laplace transform to Eq. (2.1) in the light of Eq. (2.2) we get

$$s^n\bar{u}(s)-s^{n-1}u(0)-s^{n-2}u'(0)-s^{n-3}u''(0)=s^{n-4}u'''(0)=\cdots\cdots=u^{n-1}(0)=-\bar{y}(s), \quad (5)$$

Where $\bar{u}(s)$ and $\bar{y}(s)$ is the Laplace transform of $u(t)$ and $y(t)$ respectively.
The Laplace inversion of Eq. (5) gives final solution as:

$$u(t)=\int_0^1 \frac{t^{n-1}(1-s)^{n-4}}{(n-1)!}y(s)ds-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)ds. \quad (6)$$

The proof is complete.

It is obvious that

$$G(t,s)\ge 0 \quad \text{and} \quad G(1,s)\ge G(t,s), \qquad 0\le t,s\le 1. \quad (7)$$

Lemma 3. $G(t,s)\ge q(t)G(1,s)$ for $0\le t,\ s\le 1$, where $q(t)=t^{n-1}$.

Proof: If $t\le s$, then

$$\frac{G(t,s)}{G(1,s)}=t^{n-1}.$$

If $t\ge s$, then

$$\frac{G(t,s)}{G(1,s)}=\frac{t^{n-1}(1-s)^{n-4}-(t-s)^{n-1}}{(1-s)^{n-4}-(1-s)^{n-1}}\ge t^{n-1}$$

The proof is complete.

3. Main results

In this section, we will apply Krasnoselskii's fixed-point theorem to the eigenvalue problem (1). We note that $u(t)$ is a solution of (1), (2) if and only if

$$u(t)=\lambda\int_0^1 G(t,s)a(s)f(u(s))ds, \qquad 0\le t\le 1.$$

For our constructions, we shall consider the Banach space $X=C[0,1]$ equipped with standard norm $\|u\|=\max_{0\le t\le 1}|u(t)|, u\in X$. Define a cone P by

$$P=\{u\in X \mid u(t)\ge 0,\ u(t)\ge q(t)\|u\|,\ t\in[0,1]\}.$$

It is easy to see that if $u\in P$, then $\|u\|=u(1)$. Define an integral operator by:

$$T\,u(t)=\lambda\int_0^1 G(t,s)a(s)f(u(s))ds, \qquad 0\le t\le 1, \quad u\in P. \quad (8)$$

Lemma 4. $T(P)\subset P$.

Proof: Notice from lemma (3) that, for $u\in P$, $T\,u(t)\ge 0$ on $[0,1]$ and

$$\begin{aligned} Tu(t) &= \lambda\int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &\geq \lambda\, q(t)\int_0^1 G(1,s)a(s)f(u(s))\,ds \\ &\geq \lambda\, q(t)\max_{0\leq t\leq 1}\int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &= q(t)\,\|Tu(t)\|, \quad \text{for all } t,s\in[0,1]. \end{aligned}$$

Thus $T(P)\subset P$.

By standard argument, it is easy to see that $T: P\to P$ is a completely continuous.
Following Sun and Wen [4], we define some important constants:

$$A=\int_0^1 G(1,s)\,a(s)\,q(s)\,ds, \qquad B=\int_0^1 G(1,s)\,a(s)\,ds,$$

$$F_0=\limsup_{u\to 0^+}\frac{f(u)}{u}, \qquad f_0=\liminf_{u\to 0^+}\frac{f(u)}{u}.$$

$$F_\infty=\limsup_{u\to +\infty}\frac{f(u)}{u}, \qquad f_\infty=\liminf_{u\to +\infty}\frac{f(u)}{u}$$

Here we assume that $\frac{1}{A f_\infty}=0$ if $f_\infty\to\infty$ and $\frac{1}{B F_0}=\infty$ if $F_0\to 0$ and $\frac{1}{A f_0}=0$ if $f_0\to\infty$ and $\frac{1}{B F_\infty}=\infty$ if $F_\infty\to 0$.

Theorem 1. Suppose that $A f_\infty > B F_0$. Then for each $\lambda\in\left(\frac{1}{A f_\infty},\frac{1}{B F_0}\right)$ the problem (1) and (2) has at least one positive solution.

Proof: Following Sun and Wen [4], we choose $\varepsilon>0$ sufficiently small such that $(F_0+\varepsilon)\lambda B\leq 1$. Then we have $\|Tu\|\leq u$. By definition of F_0, we see that there exists an $l_1>0$, such that $f(u)\leq (F_0+\varepsilon)u$ for $0<u\leq l_1$. If $u\in P$ with $\|u\|=l_1$, we have

$$\begin{aligned} \|Tu\| = (Tu)(1) &= \lambda\int_0^1 G(1,s)\,a(s)\,f(u(s))\,ds \\ &= \lambda\int_0^1 G(1,s)\,a(s)(F_0+\varepsilon)u(s)\,ds \\ &\leq \lambda(F_0+\varepsilon)\|u\|\int_0^1 G(1,s)\,a(s)\,ds \\ &\leq \lambda B(F_0+\varepsilon)\|u\|\leq \|u\|. \end{aligned}$$

Then we have $\|Tu\| \le \|u\|$. Thus if we let $\Omega_1 = \{u \in X \mid \|u\| < l_1\}$, then $\|Tu\| \le \|u\|$ for $u \in P \cap \partial\Omega_1$.

Following Yang [5], we choose $\varepsilon > 0$ and $c \in (0, \frac{1}{4})$, such that

$$\lambda\left((f_\infty - \varepsilon)\int_c^1 G(1,s)\,a(s)\,q(s)\,ds \right) \ge 1.$$

There exists $l_3 > 0$, such that $f(u) \ge (f_\infty - \varepsilon)u$ for $u > l_3$. Let $l_2 = \max\left\{\frac{l_3}{q(c)}, 2l_1\right\}$. If $u \in P$ with $\|u\| = l_2$, then we have

$$u(t) \ge q(t)l_2 \ge q(c)l_2 \ge l_3.$$

Therefore, for each If $u \in P$ with $\|u\| = l_2$, we have

$$\|Tu\| = (Tu)(1) = \lambda\int_0^1 G(1,s)\,a(s)\,f(u(s))\,ds$$

$$\ge \lambda\int_0^1 G(1,s)\,a(s)\,(f_\infty - \varepsilon)u(s)\,ds$$

$$\ge \lambda(f_\infty - \varepsilon)\|u\|\int_0^1 G(1,s)\,a(s)\,q(s)\,ds \ge \|u\|.$$

Thus if we let $\Omega_2 = \{u \in X \mid \|u\| < l_2\}$, then $\Omega_1 \subset \overline{\Omega}_2$ and $\|Tu\| \ge \|u\|$ for $u \in P \cap \partial\Omega_2$.

Condition (H1) of Krasnoselskii's fixed-point theorem is satisfied. So there exists a fixed point of T in P. This completes the proof.

Theorem 2. Suppose that $A f_0 > B F_\infty$. Then for each $\lambda \in (\frac{1}{A f_0}, \frac{1}{B F_\infty})$ the problem (1) and (2) has at least one positive solution.

The Proof of theorem 2 is very similar to that of theorem 1 and therefore omitted.

Theorem 3. Suppose that $\lambda B f(u) < u$ for $u \in (0,\infty)$. Then the problem (1), (2) has no positive solution.

Proof*:* Following El- Shahed[1], assume to the contrary that u is a positive solution of (1), (2). Then

$$u(1) = \lambda\int_0^1 G(1,s)\,a(s)\,f(u(s))\,ds < \frac{1}{B}\int_0^1 G(1,s)\,a(s)\,u(s)\,ds \le \frac{u(1)}{B}\int_0^1 G(1,s)\,a(s)\,ds = u(1).$$

This is a contradiction and completes the proof.

Theorem 4. Suppose that $\lambda A f(u) > u$ for $u \in (0,\infty)$. Then the problem (1), (2) has no positive solution.
The Proof of theorem 4 is very similar to that of theorem 3 and therefore omitted.

Example 1. Consider the equation

$$u^{(4)}(t) + \lambda \frac{t}{2} \frac{6u^2 + u}{u+4}(6 + \sin u) = 0, \quad 0 \le \mathrm{t} \le 1 \tag{9}$$

$$u(0) = u'(0) = u''(0), \quad \mathrm{u}'''(1) = 0 .$$

(10) Then $F_0 = f_0 = 6,\ F_\infty = 42,\ f_\infty = 30$ and $6u < f(u) < 42u$. By direct calculations, we obtain that $A = 0.016369$ and $B = 0.0375$. From theorem 1 we see that if $\lambda \in (2.0363636, 4.44444)$, then the problem (9), (10) has a positive solution. From theorem 3 we have that if $\lambda < 0.6349206$, then the problem (9), (10) has no positive solution. By theorem 4 we have that if $\lambda > 10.181818$, then the problem (9), (10) has no positive solution.

REFERENCES

[1] El-Shahed, M. (2009), Positive solutions for nonlinear singular third order boundary value problem, Communications in Nonlinear Science and Numerical Simulations. 14 424-429.

[2] Guo, D and Lakshmikantham, V. 1988, Nonlinear problems in abstract cones, Academic Press, San Diego.

[3] Li, S. 2006, Positive solutions of nonlinear singular third-order two-point boundary value problem, J. Math. Anal. Appl., 323,413–425.

[4] Sun, H. and Wen, W. 2006, On the Number of Positive solutions for a nonlinear third order boundary value problem, International Journal of Difference Equations, 1,165-176.

[5] Yang, B. 2005, Positive solutions for a fourth order boundary value problem, Electronic Journal of Qualitative Theory of Differential Equations,3, 1-17.

Received: January, 2010