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Optimal Sequential Procedures with Bayes Decision Rules

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Abstract

In this article, a general problem of sequential statistical inference for general discrete-time stochastic processes is considered. Let $X_1, X_2, \dots$ be a discrete-time stochastic process, whose distribution depends on an unknown parameter θ, $\theta \in \Theta$. We consider a problem of optimal sequential decision-making in the following framework. Let $w_n(\theta, d; x_1, \dots, x_n)$, $\theta \in \Theta$, $d \in \mathrm{D}$, be a loss function representing losses from making a decision d at stage n of a statistical experiment, when the true parameter value is θ, and the data observed up to this stage are $x_1, \dots, x_n$. Let $K_\theta^n(x_1, \dots, x_n)$ be the cost of the observations when θ is the true value of the parameter. The decision is supposed to be taken through a sequential decision-making procedure (τ, δ), where τ is a stopping time with respect to the sequence of σ-algebras $\mathrm{F}_n = \sigma(X_1, X_2, \dots, X_n)$, $n = 1, 2, \dots$, and δ is an F_τ-measurable decision function with values in D. For any sequential decision procedure (τ, δ) let us define the average loss due to incorrect decision

$$W(\theta; \tau, \delta) = E_\theta w_\tau(\theta, \delta; X_1, \dots X_\tau),$$

and the average cost of observations as

$$C(\theta; \tau) = E_\theta K_\theta^\tau(X_1, \dots, X_\tau).$$

Let, finally, the "risk function" be defined as

$$R(\tau, \delta) = \int_\Theta W(\theta; \tau, \delta) d\pi_1(\theta) + \int_\Theta C(\theta; \tau) d\pi_2(\theta),$$

where π_1 and π_2 are some probability measures on Θ. The main goal of this article is to give conditions of existence of sequential decision procedures which minimize $R(\tau, \delta)$ (optimal decision procedures), and characterize their structure. In particular, when $\pi_1 = \pi_2 = \pi$ is an *a priori* distribution of the parameter, we give a characterization of optimal (Bayesisan) sequential decision procedures minimizing $R(\tau, \delta)$ among all sequential decision procedures (τ, δ).

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1 Introduction

Let $X_1, X_2, \ldots, X_n, \ldots$ be a discrete-time stochastic process, whose distribution depends on an unknown "parameter" θ, $\theta \in \Theta$. In this article, we consider a general problem of sequential statistical decision making based on the observations of this process.

Let us define a *sequential statistical procedure* as a pair (ψ, δ), being ψ a (randomized) *stopping rule*, $\psi = (\psi_1, \psi_2, \ldots, \psi_n, \ldots)$, and δ a (terminal) *decision function*, $\delta = (\delta_1, \delta_2, \ldots, \delta_n, \ldots)$, supposing that $\psi_n = \psi_n(x_1, x_2, \ldots, x_n)$ and $\delta_n = \delta_n(x_1, x_2, \ldots, x_n)$ are measurable functions, and $\psi_n(x_1, \ldots, x_n) \in [0, 1]$, $\delta_n(x_1, \ldots, x_n) \in \mathrm{D}$ for any observations vector $(x_1, \ldots, x_n)$, for any $n = 1, 2, \ldots$ (see, for example, [10], [1], [3]).

For any stage number $n \geq 1$, $\psi_n(x_1, \ldots, x_n)$ is interpreted as the conditional probability to stop and proceed to decision making, given that we did not stop before and that the observations up to this stage were $(x_1, \ldots, x_n)$, and $\delta_n(x_1, \ldots, x_n)$ as the decision to take when stopping occurs, $n = 1, 2, \ldots$.

The stopping rule ψ generates a random variable τ_ψ (*stopping time*) whose distribution is given by

$$P_\theta(\tau_\psi = n) = E_\theta(1 - \psi_1)(1 - \psi_2) \ldots (1 - \psi_{n-1})\psi_n, \quad n = 1, 2, \ldots \tag{1}$$

(here, and in what follows, we interchangeably use ψ_n both for $\psi_n(x_1, x_1, \ldots, x_n)$ and for $\psi_n(X_1, X_1, \ldots, X_n)$: it ψ_n is under the expectation or probability sign, then it is $\psi_n(X_1, \ldots, X_n)$, otherwise it is $\psi_n(x_1, \ldots, x_n)$).

Let $w_n(\theta, d; x_1, \ldots, x_n)$ be a non-negative loss function, $n = 1, 2, \ldots$ (we suppose that w_n is a measurable function of all its arguments for any $n \geq 1$). Let π_1 be any probability measure. We define the average loss of the sequential statistical procedure (ψ, δ) due to wrong decision as

$$W(\psi, \delta) = \sum_{n=1}^{\infty} \int \left[E_\theta(1 - \psi_1) \ldots (1 - \psi_{n-1})\psi_n w_n(\theta, \delta_n; X_1, \ldots, X_n)\right] d\pi_1(\theta). \tag{2}$$

Let also $K_\theta^n = K_\theta^n(x_1, \dots, x_n)$ be a non-negative (and measurable with respect to $(\theta, x_1, \dots, x_n)$) cost function, $n \geq 1$, such that $K_\theta^n(x_1, \dots, x_n) \leq K_\theta^{n+1}(x_1, \dots, x_n, x_{n+1})$ for any observation sequence $x_1, x_2, \dots, x_{n+1}$, $n \geq 1$, $\theta \in \Theta$.

Let us define the *average cost* of the sequential decision procedure (τ, δ) as

$$C(\theta; \psi) = E_\theta K_\theta^{\tau_\psi}(X_1, \dots, X_{\tau_\psi})$$

(we suppose that $C(\theta; \psi) = \infty$ if $\sum_{n=1}^{\infty} P_\theta(\tau_\psi = n) < 1$, see (1)).

Let us also define a "weighted" value of the average cost

$$C(\psi) = \int C(\theta; \psi) d\pi_2(\theta), \tag{3}$$

where π_2 is some probability measure giving "weights" to particular values of θ.

Our main goal is minimizing the "weighted risk"

$$R(\psi, \delta) = C(\psi) + W(\psi, \delta), \tag{4}$$

supposing that π_1 in (2) and π_2 in (3) are, generally speaking, two *different* probability measures. If $\pi_1 = \pi_2 = \pi$, $R(\psi, \delta)$ is called *Bayesian risk* of (ψ, δ) corresponding to the *a priori* distribution π (see, for example, [11], [10], [1], [9], [3], among many others).

To guarantee that $\inf R(\psi, \delta)$ is finite we suppose that $\inf_\delta R(\psi^1, \delta) < \infty$ with $\psi^1 = (1, \dots)$.

We use essentially the same method as in [4], where the case of $K_\theta^n \equiv n$ and $w_n(\theta, d; x_1, \dots, x_n) \equiv w(\theta, d)$ for any $\theta \in \Theta$, $d \in \mathrm{D}$, and for any $(x_1, \dots, x_n)$, $n \geq 1$, was considered. In turn, the method of [4] is an extension of the results of [6]. In [6], there is a number of applications of the results of this nature to hypothesis testing problems starting from classical problems of Wald and Wolfowitz [11] and of Kiefer-Weiss (see [12]) to Bayesian hypothesis testing problems for stochastic processes considered in [2]. The same method is used in [7] for multiple hypothesis testing problems. An extension of this method for statistical problems with control variables can be found in [5] and in [8].

2 Main results

Throughout the paper we suppose that for any $n = 1, 2, \dots,$ the vector $(X_1, X_2, \dots, X_n)$ has a probability "density" function

$$f_\theta^n = f_\theta^n(x_1, x_2, \dots, x_n)$$

(Radon-Nikodym derivative of its distribution) with respect to a product-measure

$$\mu^n \quad = \quad \underbrace{\mu \otimes \mu \otimes \cdots \otimes \mu}_{n \quad \text{times}},$$

with some σ-finite measure μ on the respective space. As usual in the Bayesian context, we suppose that $f_\theta^n(x_1, x_2, \ldots, x_n)$ is measurable with respect to $(\theta, x_1, \ldots, x_n)$, for any $n = 1, 2, \ldots$.

Let us suppose that for any $n \geq 1$ there exists a measurable $\delta_n^B = \delta_n^B(x_1, \ldots, x_n)$ such that for any $d \in \mathrm{D}$

$$\int w_n(\theta, d; x_1, \ldots, x_n) f_\theta^n(x_1, \ldots, x_n) d\pi_1(\theta)$$

$$\geq \int w_n(\theta, \delta_n^B; x_1, \ldots, x_n) f_\theta^n(x_1, \ldots, x_n) d\pi_1(\theta) \tag{5}$$

for all data sequences $(x_1, \ldots, x_n)$. Let $\delta^B = (\delta_1^B, \delta_2^B, \ldots, \delta_n^B, \ldots)$. It is easy to see that in this case for any decision function $\delta_n = \delta_n(x_1, \ldots, x_n)$

$$\int_\Theta E_\theta w_n(\theta, \delta_n; X_1, \ldots, X_n) d\pi_1(\theta) \geq \int_\Theta E_\theta w_n(\theta, \delta_n^B; X_1, \ldots, X_n) d\pi_1(\theta),$$

i.e. δ_n^B is a *Bayesian decision function* (corresponding to the "a priori" distribution π_1) based on n observations.

Let us denote $l_n = l_n(x_1, \ldots, x_n)$ the right-hand side of (5). From this time on, we suppose that $\int l_n d\mu_n < \infty$ for any $n = 1, 2, \ldots$.

In the same way as in [4] we easily get

Theorem 2.1 *For any sequential decision procedure* (ψ, δ)

$$W(\psi, \delta) \geq W(\psi, \delta^B) = \sum_{n=1}^{\infty} \int (1 - \psi_1) \ldots (1 - \psi_{n-1}) \psi_n l_n d\mu^n.$$

It follows from Theorem 2.1 that $\inf_\delta W(\psi, \delta) = W(\psi, \delta^B)$, and the aim of what follows is to minimize

$$L(\psi) = C(\psi) + W(\psi, \delta^B)$$

over all stopping rules ψ (see (4)).

It is easy to see that, by definition of $C(\psi)$,

$$L(\psi) = \sum_{n=1}^{\infty} \int (1 - \psi_1) \ldots (1 - \psi_{n-1}) \psi_n \left(\int K_\theta^n f_\theta^n d\pi_2(\theta) + l_n \right) d\mu^n \tag{6}$$

if $\int P_\theta(\tau_\psi < \infty) d\pi_2(\theta) = 1$, and $L(\psi) = \infty$ otherwise.

Let us denote

$$k_n = k_n(x_1, \ldots, x_n) = \int K_\theta^n(x_1, \ldots, x_n) f_\theta^n(x_1, \ldots, x_n) d\pi_2(\theta)$$

(see (6)), and let for any $\pi = \pi_1$ or $\pi = \pi_2$ $P^\pi(A) = \int P_\theta(A) d\pi(\theta)$ for any event A.

Let also

$$s_n^\psi = s_n^\psi(x_1, \ldots, x_n) = (1 - \psi_1(x_1)) \ldots (1 - \psi_{n-1}(x_1, \ldots, x_{n-1}))\psi_n(x_1, \ldots, x_n)$$

for any $n = 1, 2, \ldots$ and for any stopping rule ψ.

Thus, by (6),

$$L(\psi) = \sum_{n=1}^{\infty} \int s_n^\psi \, (k_n + l_n) \, d\mu^n$$

if $P^{\pi_2}(\tau_\psi < \infty) = 1$, and $L(\psi) = \infty$ otherwise.

First, let us solve the problem of minimization of $L(\psi)$ in the class F^N of truncated stopping rules, that is such that $\psi = (\psi_1, \psi_2, \ldots, \psi_{N-1}, 1, \ldots)$, $N = 1, 2, \ldots$ (see also [4]).

For any $\psi \in \mathrm{F}^N$ let

$$L_N(\psi) = \sum_{n=1}^{N} \int s_n^\psi \, (k_n + l_n) \, d\mu^n = \sum_{n=1}^{N-1} \int s_n^\psi \, (k_n + l_n) \, d\mu^n + \int t_N^\psi \, (k_N + l_N) \, d\mu^N,$$

where, for any $n \geq 1$ and for any stopping rule ψ

$$t_n^\psi = t_n^\psi(x_1, \ldots, x_n) = (1 - \psi_1(x_1)) \ldots (1 - \psi_{n-1}(x_1, \ldots, x_{n-1})), \quad t_1 \equiv 1.$$

Theorem 2.2 *Let $\psi \in \mathrm{F}^N$ be any (truncated) stopping rule, $N \geq 2$. Then for any $1 \leq r \leq N-1$ the following inequalities hold true*

$$L_N(\psi) \geq \sum_{n=1}^{r} \int s_n^\psi (k_n + l_n) d\mu^n + \int t_{r+1}^\psi \left(k_{r+1} + V_{r+1}^N\right) d\mu^{r+1} \tag{7}$$

$$\geq \sum_{n=1}^{r-1} \int s_n^\psi (k_n + l_n) d\mu^n + \int t_r^\psi \left(k_r + V_r^N\right) d\mu^r, \tag{8}$$

where $V_N^N \equiv l_N$, and recursively for $m = N-1, N-2, \ldots 1$

$$V_m^N = \min\{l_m, Q_m^N\}, \tag{9}$$

where

$$Q_m^N = \int \left(k_{m+1} + V_{m+1}^N\right) d\mu(x_{m+1}) - k_m \tag{10}$$

(it should be remembered that the function under the integral sign on the right-hand side of (10) is a function of $(x_1, \dots, x_{m+1})$*, and, because of this,* $Q_m^N = Q_m^N(x_1, \dots, x_m)$*).*

The lower bound in (8) is attained if and only if

$$I_{\{l_m < Q_m^N\}} \leq \psi_m \leq I_{\{l_m \leq Q_m^N\}} \tag{11}$$

μ^m*-almost everywhere on*

$$T_m^\psi = \{(x_1, \dots, x_m) : t_m^\psi(x_1, \dots, x_m) > 0\},$$

for all $m = r, r+1, \dots, N-1$*.*

In particular, $(\psi_1, \psi_2, \dots, \psi_{N-1}, 1, \dots)$ *is an optimal truncated stopping rule in* F^N*, if and only if (11) is satisfied* μ^m*-almost everywhere on* T_m^ψ *for all* $m = 1, \dots, N-1$*. In addition,*

$$\inf_{\psi \in \mathrm{F}^N} L(\psi) = Q_0^N, \tag{12}$$

where

$$Q_0^N = \int \left(k_1(x) + V_1^N(x)\right) d\mu(x).$$

Proof. The proof can be implemented by induction as in the proof of Theorem 3 in [4] using instead of Lemma 2 [4] the following extension of it.

Lemma 2.3 *Let* $r \geq 1$ *be any natural number, and let* $v_{r+1} = v_{r+1}(x_1, x_2, \dots, x_{r+1})$ *be any non-negative measurable function, such that* $\int v_{r+1} d\mu^{r+1} < \infty$*. Then*

$$\int s_r^\psi (k_r + l_r) d\mu^r + \int t_{r+1}^\psi \left(k_{r+1} + v_{r+1}\right) d\mu^{r+1} \geq \int t_r^\psi \left(k_r + v_r\right) d\mu^r, \tag{13}$$

where $v_r = \min\{l_r, Q_r\}$*, with* $Q_r = Q_r(x_1, \dots, x_r)$ *defined as*

$$Q_r(x_1, \dots, x_r) = \int \left(k_{r+1}(x_1, \dots, x_{r+1}) + v_{r+1}(x_1, \dots, x_{r+1})\right) d\mu(x_{r+1})$$

$$-k_r(x_1, \dots, x_r).$$

There is an equality in (13) if and only if $I_{\{l_r < Q_r\}} \leq \psi_r \leq I_{\{l_r \leq Q_r\}}$ μ^r*-almost everywhere on* T_r^ψ*.*

Proof of Lemma 2.3 can be implemented following the steps of the proof of Lemma 2 in [4] and is omitted here. ■

Starting with the class of non-truncated stopping rule, let us define for any ψ

$$L_N(\psi) = \sum_{n=1}^{N-1} \int s_n^{\psi}(k_n + l_n)\, d\mu^n + \int t_N^{\psi}(k_N + l_N)\, d\mu^N.$$

The idea of construction of optimal stopping rules is to pass to the limit, as $N \to \infty$, in (7), (8), (9) and (10).

Let F be a class of stopping rules such that for every $\psi \in \mathrm{F}$

$$P^{\pi_2}(\tau_\psi < \infty) = 1 \quad \text{and} \quad \lim_{N\to\infty} L_N(\psi) = L(\psi).$$

In a very similar manner as in [4] it can be shown that for any $m = 1, 2, \ldots$ y any $N \geq m$ $V_m^N(x_1, \ldots, x_m) \geq V_m^{N+1}(x_1, \ldots, x_m)$ for any $(x_1, \ldots, x_m)$, so there exists

$$V_m = V_m(x_1, \ldots, x_m) = \lim_{N\to\infty} V_m^N(x_1, \ldots, x_m).$$

Thus, passing to the limit, for any $\psi \in \mathrm{F}$, in (7), (8), (9) and (10) is justified by the Lebesgue's dominated convergence theorem. In particular, let

$$Q_m = Q_m(x_1, \ldots, x_m) = \lim_{N\to\infty} Q_m^N(x_1, \ldots, x_m), \quad m = 0, 1, 2, \ldots.$$

In the same way as in [4] it can be shown that (cf. (12))

$$\inf_{\psi\in\mathrm{F}} L(\psi) = Q_0 = \int (k_1(x) + V_1(x))\, d\mu(x).$$

Combining all these ideas, we immediately have

Theorem 2.4 *If there exists $\psi \in \mathrm{F}$ such that*

$$L(\psi) = \inf_{\psi'\in\mathrm{F}} L(\psi') \tag{14}$$

then

$$I_{\{l_m < Q_m\}} \leq \psi_m \leq I_{\{l_m \leq Q_m\}} \tag{15}$$

μ^m-almost everywhere on T_m^{ψ}, for all $m = 1, 2, \ldots$.

On the other hand, if ψ satisfies (15) μ^m-almost everywhere on T_m^{ψ}, for any $m = 1, 2, \ldots$, and $\psi \in \mathrm{F}$, then it satisfies (14) as well.

Proof. The proof can be conducted following the steps of the proof of Theorem 4 in [4], using Lemma 2.3 instead of Lemma 2 of [4]. ■

Very much like in [4], we can give some conditions, under which the structure of (15) is necessary and sufficient for optimality in the class of all stopping rules.

Let us call the problem of minimizing $L(\psi)$ *truncatable* if for any ψ such that $P^{\pi_2}(\tau_\psi < \infty) = 1$ it holds $L_N(\psi) \to L(\psi)$, as $N \to \infty$.

Theorem 2.5 *Let the problem of minimizing $L(\psi)$ be truncatable, and let for any $c > 0$*

$$\int P_\theta(K_\theta^n(X_1,\ldots,X_n) < c)d\pi_2(\theta) \to 0 \quad as \quad n \to \infty. \tag{16}$$

Then

$$L(\psi) = \inf_{\psi'} L(\psi')$$

if and only if

$$I_{\{l_m < Q_m\}} \le \psi_m \le I_{\{l_m \le Q_m\}} \tag{17}$$

μ^m-almost everywhere on T_m^ψ, for all $m = 1, 2, \ldots$.

Proof. The "if"-part can be proved analogously to the proof of Theorem 4 in [4], using Lemma 2.3 instead of Lemma 2 in [4].

To prove the "only if"-part we suppose that ψ satisfies (15) μ^m-almost everywhere on T_m^ψ, for any $m = 1, 2, \ldots$. It follows from Lemma 2.3 that for any fixed $m = 1, 2, \ldots$

$$\sum_{n=1}^{m-1} \int s_n^\psi(k_n + l_n)d\mu^n + \int t_m^\psi(k_m + V_m)d\mu^m = \int (k_1(x) + V_1(x))d\mu(x) = I < \infty. \tag{18}$$

In particular, this implies that $\int t_m^\psi k_m d\mu^m \le I$, or

$$\int E_\theta t_m^\psi K_\theta^m d\pi_2(\theta) \le I, \tag{19}$$

where $t_m^\psi = t_m^\psi(X_1, \ldots, X_m)$ and $K_\theta^m = K_\theta^m(X_1, \ldots, X_m)$.

Let C be any positive constant. Then (19) implies

$$C \int E_\theta t_m^\psi I_{\{K_\theta^m > C\}} d\pi_2(\theta) < I, \quad m = 1, 2, \ldots. \tag{20}$$

Because

$$\int E_\theta t_m^\psi d\pi_2(\theta) = \int E_\theta t_m^\psi I_{\{K_\theta^m > C\}} d\pi_2(\theta) + \int E_\theta t_m^\psi I_{\{K_\theta^m \le C\}} d\pi_2(\theta) \tag{21}$$

and the second summand by virtue of (16) tends to 0, as $m \to \infty$, we have that the difference between the first summand on the right-hand side of (21) and the left-hand side of it, goes to 0 as $m \to \infty$. Thus, from (20), we have that

$$\lim_{m\to\infty} \int E_\theta t_m^\psi d\pi_2(\theta) = \lim_{m\to\infty} \int P_\theta(\tau_\psi \geq m) d\pi_2(\theta) = \int P_\theta(\tau_\psi = \infty) d\pi_2(\theta) < I/C,$$

and, because of arbitrariness of C, $P^{\pi_2}(\tau = \infty) = 0$, or

$$P^{\pi_2}(\tau < \infty) = 1. \tag{22}$$

Now, from (18) we get that

$$\lim_{m\to\infty} \sum_{n=1}^{m-1} \int s_n^\psi (k_n + l_n) d\mu^n = L(\psi) \leq I. \tag{23}$$

Because the problem is truncatable, it follows from (22) that $L_N(\psi) \to L(\psi)$, as $N \to \infty$. Now, passing to the limit in (12), we get $L(\psi) \geq I$. From this and (23) it follows that $L(\psi) = I = \inf_{\psi'} L(\psi')$. ∎

Very much like in [4] (see Corollary 1 therein), there are simple conditions which guarantee that the problem is truncatable.

Theorem 2.6 *The problem of minimization of $L(\psi)$ is truncatable if any of the following two conditions is fulfilled.*

(i) *There is M, $0 < M < \infty$, such that $w_n(\theta, d; x_1, \ldots, x_n) \leq M$ for any $\theta, d, x_1, \ldots, x_n$, and for any $n \geq 1$, and from $L(\psi) < \infty$ it follows that*

$$P^{\pi_1}(\tau_\psi < \infty) = 1.$$

(ii)

$$\int l_n d\mu^n \to 0, \quad \text{as} \quad n \to \infty.$$

Theorem 2.6 can be proved in the same way as Corollary 1 in [4].

Combining Theorem 2.1 with Theorem 2.2 or Theorem 2.4 or Theorem 2.5, we have, under respective conditions, sequential decision procedures (ψ, δ^B) minimizing $R(\psi, \delta)$ in the corresponding class of sequential decision procedures, and the respective necessary conditions under which the minimum is attained, for example, using Theorem 2.5 we get:

Theorem 2.7 *Under the conditions of Theorem 2.5*

$$\inf_{(\psi,\delta)} R(\psi, \delta) = Q_0.$$

For every ψ satisfying (17) (μ^m-almost everywhere on T_m^ψ) for all $m = 1, 2, \ldots$ it holds

$$R(\psi, \delta^B) = Q_0.$$

If for a sequential decision procedure (ψ, δ) $R(\psi, \delta) = Q_0$, then ψ satisfies (17) μ^m-almost everywhere on T_m^ψ for all $m = 1, 2, \ldots$.

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