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Taylor Transformations into G²

Mulatu Lemma

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Abstract. Through out this paper, we assume that $0 < r$. The Taylor transformations T_r are given by the positive, regular matrix defined by

$$a_{nk} = \binom{k}{n} r^{k-n} (1-r)^{n+1} \qquad (n, k = 0,1,2,....)$$

The purpose of this paper is to study these matrices as mappings into G^2. The necessary and sufficient conditions for T_r to be $G^2 - G^2$ is proved. The strength of T_r in the $G^2 - G^2$ setting is investigated. Also, It is shown that every $G^2 - G^2$ T_r matrix is $G^2 -$translative.

1. Introduction and Background.

The Taylor transformations are studied as $G - G$ mappings in [4]. So, it a natural question to ask if there is a theory for T_r in the $G^2 - G^2$ setting that parallels the theory of T_r in the $c - c$ setting.

The answer is affirmative, and it produces this paper.

2. Basic Notations and Definitions.

Let $A = (a_{nk})$ be an infinite matrix defining a sequence to sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \qquad (2.1)$$

where $(Ax)_n$ denotes the nth term of the image sequence *Ax*. The sequence *Ax* is called the *A*-transform of the sequence *x*.

Let *y* be a complex number sequence. Throughout this paper, we shall use the following basic notations.

C = { set of all convergent sequences}

$G = \left\{ y : y_k = O\left(r^k\right) \text{ for some } r \in (0,1) \right\}$

$G^2 = \left\{ y : \left(y_k\right)^2 = o\left(r^k\right) \text{ for some } r \in (0,1) \right\}$

$G^2(A) = \left\{ y : Ay \in G^2 \right\}$

Definition 1. If *X* and *Y* are complex number sequences, then the matrix *A* is called an *X*-*Y* matrix if the image Au of u under the transformation *A* is in *Y* whenever *u* is in *X*.

Definition 2. The summability matrix *A* is said to be G^2- translative for a sequence u in $G^2(A)$ provided that each of the sequences Y_u and Z_u is in $G^2(A)$, where $Y_u = \{u_1, u_2, u_3, ...\}$ and $Z_u = \{0, u_0, u_1, ...\}$.

Definition 3. The matrix *A* is G^2-stronger than the matrix *B* provided that

$$G^2(B) \subset G^2(A) \qquad .$$

3. The Main Results.

Our first theorem deals with the necessary and sufficient conditions for T_r to be a $G^2 - G^2$ matrix.

Remark 1: Note that if

$x \in G^2$, then $|x_k| \le Mp^k$ for some $p \in (0,1)$

Lemma 1. If T_r is a $G^2 - G^2$ matrix, then $0 < r \le 1$

Lemma 2. If $0 < r \le 1$, then T_r is a $G^2 - G^2$ matrix .

Proof. We will show that $x \in G^2$ implies that $T_r x \in G^2$. Note that $x \in G^2$ implies that $|x_k| \le Mp^k$ for some $M > 0$ and $p \in (0,1)$.

$$\left|(T_r x)_n\right| = \left|(1-r)^{n+1} \sum_{k=n}^{\infty} \binom{k}{n} r^{k-n} x_k\right|$$

$$= \left|(1-r)^{n+1} \sum_{k=0}^{\infty} \binom{k+n}{k} r^k x_{k+n}\right|$$

$$\le Mp^n (1-r)^{n+1} \left|\sum_{k=0}^{\infty} \binom{k+n}{k} (rp)^k\right|$$

$$= Mp^n (1-r)^{n+1} (1+rp)^{-(n+1)}$$

$$< Mp^n.$$

Hence, $T_r x \in G^2$ and thus T_r is a matrix. $G^2 - G^2$

Theorem 1. T_r is a $G^2 - G^2$ matrix if and only if $0 < r \le 1$

Proof The theorem follows by Lemmas 1 & 2.

Remark 1. The fact that T_r is not a $G^2 - G^2$ matrix for $r > 1$ can also be justified by the following example.

Example . Suppose that $r = 3$ and let x be a sequence such that $x_k = \left(\frac{1}{4}\right)^k$.

Then we will show that the sequence $T_3 x$ is not in G^2.

$$\left|(T_3 x)_n\right| = \left|(1-r)^{n+1} \sum_{k=n}^{\infty} \binom{k}{n} 3^{k-n} x_k\right|$$

$$= \left(\tfrac{1}{4}\right)^n (2)^{n+1} \left|\sum_{k=0}^{\infty} \binom{k+n}{k} \left(\tfrac{3}{4}\right)^k\right|$$

$$= \left| 2^{n+1} \sum_{k=0}^{\infty} \binom{k+n}{k} 3^k \, {}^1\!/\!{}_4{}^k \right| \tag{3.2}$$

$$= 2\left({}^1\!/\!{}_2\right)^n \left({}^1\!/\!{}_4\right)^{-(n+1)}$$

$$< 8(2)^n.$$

Hence, it follows that $T_3 x$ is not in G^2**.** Thus, T_3 is not a $G^2 - G^2$.

Theoerm 2. Suppose $r \neq 1$ and $x_k = r^k$. Then T_r a $G^2 - G^2$ matrix implies that $\arcsin x \in G^2$.

Proof. The theorem easily follows using Theorem 1 and the inequality

$$x < \arcsin x < \frac{x}{\sqrt{1-x^2}} \quad \text{for } 0 < x < 1$$

Lemma 3. If T_3 a $G^2 - G^2$, then $G^2(T_r)$ contains a bounded (non-convergent) sequence.

Proof. Let $x_k = (-1)^k$. We will show that $x \in G^2(T_r)$. Note that

$$\left|(T_r x)_n\right| = \left| (1-r)^{n+1} \sum_{k=n}^{\infty} (\text{kn}) \, r^{k-n} x_k \right|$$

$$= (1-r)^{n+1} \left| \sum_{k=0}^{\infty} \binom{k+n}{k} r^k (-1)^{k+n} \right| \tag{3.3}$$

$$= (1-r)^{n+1} \left| \sum_{k=0}^{\infty} \binom{k+n}{k} (-r)^k \right|$$

$$= (1-r)^{n+1} (1+r)^{-(n+1)}$$

$$<\left(1-r\right)^n.$$

Hence $x\in G^2\left(T_r\right)$ if T_r a G^2-G^2 matrix

Lemma 4. If T_r a G^2-G^2, then $G^2\left(T_r\right)$ contains an unbounded sequence.

Proof. Let $1<p<3$, $x_k=\left(-p\right)^k$, and $\left(p-1\right)/2p<r<1/p$. We will show that

$x\in G^2\left(T_r\right)$. We have

$$\left|\left(T_r x\right)_n\right| = \left| \left(1-r\right)^{n+1} \sum_{k=n}^{\infty} \binom{k}{n} r^{k-n} x_k \right|$$

$$= \left|\left(1-r\right)^{n+1} \sum_{k=0}^{\infty} \binom{k+n}{k} \left(r\right)^k \left(-p\right)^{k+n}\right|$$

$$= p^n \left(1-r\right)^{n+1} \left|\sum_{k=0}^{\infty} \binom{k+n}{k} \left(-rp\right)^k\right|$$

$$= p^n \left(1-r\right)^{n+1} \left(1+rp\right)^{-(n+1)}$$

$$< \left(s\right)^n \quad ,$$

where $s = p^n \left(1-r\right)^n \left(1+rp\right)^{-n}$. Now the hypothesis that T_r a $G-G$ matrix and $\left(p-1\right)/2p<r$ implies that $s<1$. Hence, it follows that $x\in G^2\left(T_r\right)$.

Theorem 3. The T_r matrix is stronger than the identity matrix in the G^2-G^2 setting.

Proof. The theorem follows by Lemmas 3 & 4.

Lemma 5. Suppose $A=\left[a_{nk}\right]$ is a
G^2-G^2

matrix such that $a_{nk} = 0$ for $k < n, m > p$ (both positive integers); then $G^2\left(A^p\right) \subseteq G^2\left(A^m\right)$, where the interpretation for A^p and A^m is as given in [3, p. 28]

Theorem 3. If T_r is a $G^2 - G^2$ matrix, T_r^m is also a $G^2 - G^2$ matrix (for *m* a positive integer greater than)

Proof. Let $x \in G$. T_r is a $G^2 - G^2$ matrix implies that $x \in G^2\left(T_r\right)$. By Lemma 5, we have $G^2\left(T_r\right) \subseteq G^2 - G^2\left(T_r^m\right)$ and hence it follows that $x \in G^2\left(T_r^m\right)$. Thus, T_r^m is a $G^2 - G^2$

The next main result suggests that the T_r matrix is $G^2 -$ translative in the $G^2 - G^2$ setting.

Theorem 4. Every $G^2 - G^2$ T_r matrix is $G^2 -$ translative

Proof. Let $x \in G^2\left(T_r\right)$. Then we will show that:

(1) $Y_x \in G(T_r)$ and

(2) $Z_x \in G(T_r)$,

Where Y_x and Z_x are as definition 2. Let us first show (1) holds.

(3.5)

$$\left|\left(T_r Y_x\right)_n\right| = \left|(1-r)^{n+1} \sum_{k=n}^{\infty} \binom{k}{n} r^{k-n} x_{k+1}\right|$$

$$= \left|(1-r)^{n+1} \sum_{k=0}^{\infty} \binom{k+n}{k} r^k x_{k+n+1}\right|$$

$$= \frac{(1-r)^{n+1}}{r} \left|\sum_{k=1}^{\infty} \binom{k-1+n}{k-1} x_{k+n} r^k\right|$$

$$=\left(\frac{(1-r)^{n+1}}{r}\right)\left|\sum_{k=1}^{\infty}\binom{k+n}{k}x_{k+n}r^k\,\frac{k}{k+n}\right|$$

$$=\left(\frac{(1-r)^{n+1}}{r}\right)\left|\sum_{k=1}^{\infty}\binom{k+n}{k}x_{k+n}r^k\left(1-\frac{n}{k+n}\right)\right|$$

$$\le A_n+B_n \tag{3.6}$$

where

and $A_n=\dfrac{(1-r)^{n+1}}{r}\left|\sum_{k=1}^{\infty}\binom{k+n}{k}x_{k+n}r^k\right|$

$$B_n=\frac{n(1-r)^{n+1}}{r}\left|\sum_{k=1}^{\infty}\binom{k+n}{k}x_{k+n}r^k\,\frac{1}{k+n}\right| \tag{3.7}$$

Now if we show both A and B are in G^2 , then (1) holds. The condition that that $A\in G^2$ follows from the hypothesis that $x\in G^2(T_r)$, and $B\in G^2$ will be shown as follows: Observe that

(3.9)

$$B_n=\frac{n(1-r)^{n+1}}{r}\left|\sum_{k=1}^{\infty}\binom{k+n}{k}\frac{x_{k+n}}{k+n}r^k\right|$$

$$=\frac{n(1-r)^{n+1}}{r^{n+1}}\left|\sum_{k=1}^{\infty}\binom{k+n}{k}x_{k+n}\int_0^r t^{k+n-1}dt\right|$$

$$=\frac{n(1-r)^{n+1}}{r^{n+1}}\left|\int_0^r dt\left(\sum_{k=1}^{\infty}\binom{k+n}{k}x_{k+n}t^{k+n-1}\right)\right|$$

$$=\frac{n(1-r)^{n+1}}{r^{n+1}}\left|\int_0^r t^{n-1}dt\left(\sum_{k=1}^{\infty}\binom{k+n}{k}x_{k+n}t^k\right)\right|$$

Let $M_n=\left|\left(\sum_{k=1}^{\infty}\binom{k+n}{k}x_{k+n}t^k\right)\right|$.

Note that $0 < t < r$ and $x \in G^2(T_r)$ implies that $M \in G^2$. Observe that 3.8, * and 3.9 implies that

$$B_n = \frac{M_n (1-r)^{n+1}}{r} \tag{3.10}$$

Now $M \in G^2$ and $(1-r) \in G^2$ implies that $B \in G^2$. Thus, (1) holds.

Next we will show that (2) holds.

$$\left|(T_r Z_x)_n\right| = \left|(1-r)^{n+1} \sum_{k=n+1}^{\infty} \binom{k}{r} r^{k-n} x_{k-1}\right|$$

$$= \left|(1-r)^{n+1} \sum_{k=0}^{\infty} \binom{k+n+1}{n} r^{k+1} x_{k+n}\right|$$

$$= r(1-r)^{n+1} \left|\sum_{k=0}^{\infty} \binom{k+n+1}{n} x_{k+n} r^k\right| \tag{3.11}$$

$$= r(1-r)^{n+1} \left|\sum_{k=0}^{\infty} \binom{k+n}{n} x_{k+n} r^k \frac{k+1}{k+n+1}\right|$$

$$= r(1-r)^{n+1} \left|\sum_{k=0}^{\infty} \binom{k+n}{n} x_{k+n} r^k \left(1 - \frac{n}{k+n+1}\right)\right|$$

$$\leq C_n + D_n$$

where

$$C_n = r(1-r)^{n+1} \left|\sum_{k=0}^{\infty} \binom{k+n}{n} x_{k+n} r^k\right| \tag{3.12}$$

and

$$D_n = nr(1-r)^{n+1} \left|\sum_{k=0}^{\infty} \binom{k+n}{n} x_{k+n} r^k \frac{1}{k+n+1}\right| \tag{3.13}$$

By Theorem 1, the hypothesis that $T_r\ G^2 - G^2$ implies that $C \in G^2$, hence there remains only to show $D \in G^2$. Note that

$$\begin{aligned} D_n &= nr(1-r)^{n+1}\left|\sum_{k=0}^{\infty}\binom{k+n}{n}\frac{x_{k+n}}{k+n+1}r^k\right| \\ &= \frac{n(1-r)^{n+1}}{r^n}\left|\sum_{k=1}^{\infty}x_{k+n}\int_0^r t^{k+n}dt\right| \qquad (3.14) \\ &= \frac{n(1-r)^{n+1}}{r^n}\left|\int_0^r dt\left(\sum_{k=1}^{\infty}\binom{k+n}{n}x_{k+n}t^{k+n}\right)\right| \\ &= \frac{n(1-r)^{n+1}}{r^n}\left|\int_0^r t^n dt\left(\sum_{k=1}^{\infty}\binom{k+n}{n}x_{k+n}t^{k+n}\right)\right| \end{aligned}$$

$$\text{Let}\quad P_n = \left|\left(\sum_{k=1}^{\infty}\binom{k+n}{k}x_{k+n}t^k\right)\right| \qquad (3.15)$$

Note that $0 \le t \le r$ and $x \in G^2(T_r)$ implies that $P \in G$. Observe that 3.14 * and 3.15 implies that

$$D_n = \frac{nrP_n(1-r)^{n+1}}{n+1} \qquad . \qquad (3.16)$$

Now $P \in G$ and $(1-r) \in G$ implies that $D \in G$. Thus, (2) holds and the theorem follows.

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