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The Existence of Weak Solutions for a Generalized Camassa-Holm Equation[1]

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Abstract

The pseudoparabolic regularization technique is employed to investigate a generalized Camassa-Holm equation. A sufficient condition which guarantees the existence of weak solutions of the equation in lower order Sobolev space H^s with $1 < s \leq \frac{3}{2}$ is derived.

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1 Introduction

Camassa and Holm [1] derived a completely integrable wave equation

$$u_t - u_{xxt} + 3uu_x + 2ku_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

where function u is the fluid velocity in the x direction or equivalently the height of the water free surface above a flat bottom. The constant k is related to the critical shallow water wave speed. Camassa and Holm [1] showed that for all k, equation (1) is integrable, and for $k = 0$, it has travelling wave solutions of the form $ce^{-|x-ct|}$, which are called peakons. Following Camassa and Holm's work, the equation has been studied extensively. The conserved quantities and an initial value problem of equation (1) were investigated in [2]. Symmetries of equation (1) were

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discussed in [3]. Kraenkel et al [4] investigated the integral perturbation properties of (1). Cooper and Shepard [5] regarded (1) as an asymptotic model to describe long gravity waves at the surface of shallow water. Li and Olver [6] established the local well-posedness in the Sobolev space $H^s(R)$ with $s > \frac{3}{2}$ for the equation and gave conditions on the initial data that lead to finite time blow-up of certain solutions.

Recently, Hakkaev and Kirchev [7] investigated the equation

$$u_t - u_{xxt} + 2ku_x + \frac{(m+2)(m+1)}{2}u^m u_x = \left(mu^{m-1}\frac{u_x^2}{2} + u^m u_{xx}\right)_x, \quad m \geq 1, \tag{2}$$

which is a generalized form of the Camassa-Holm equation (1). In [7], the local well-posedness of a Cauchy problem for equation (2) was established in Sobolev space H^s with $s > \frac{3}{2}$, and the stability and instability of the solitary waves for the equation were discussed under some suitable assumptions.

In this article, motivated by the desire to extend parts of works in [6,7], we study the following generalized Camassa-Holm equation

$$u_t - u_{xxt} + 2ku_x + au^m u_x = \left(nu^{n-1}\frac{u_x^2}{2} + u^n u_{xx}\right)_x, \tag{3}$$

where $k \neq 0, a \neq 0, m \geq 1$ and $n \geq 1$ are constants. Obviously, equation (3) reduces to equation (2) if we set $a = \frac{(m+2)(m+1)}{2}$ and $n = m$.

The objective of this work is to investigate the existence of weak solutions for Eq.(3). A sufficient condition which guarantees the existence of solutions of the equation in the lower order Sobolev space H^s with $1 < s \leq \frac{3}{2}$ is given. It is shown that when $m = n = 1$, the main result obtained in this paper becomes part of those presented in Li and Olver [6]. In fact, Hakkaev and Kirchev [7] did not study the existence of weak solutions of the Camassa-Holm equation (2) in the lower order Sobolev space H^s with $1 < s \leq \frac{3}{2}$.

For simplicity, through out this article, we let c denote any positive constant which is independent of the small positive parameter ε.

2 Well-posedness of solutions for the regularized equation

In order to study the existence of solutions for equation (3), we consider its regularized equation with an initial condition in the form

$$\begin{cases} u_t - u_{xxt} + \varepsilon u_{xxxxt} = \partial_x(-2ku - \frac{a}{m+1}u^{m+1}) + \frac{1}{n+1}\partial_x^3(u^{n+1}) - \frac{n}{2}\partial_x(u^{n-1}u_x^2), \\ u(x,0) = u_0(x), \end{cases} \tag{4}$$

where $0 < \varepsilon < \frac{1}{4}, m \geq 1$ and $n \geq 1$. Now we give the following theorem to describe the local well-posedness of solutions for problem (4).

Theorem 2.1. Let $u_0(x) \in H^s(R)$ with $s \geq 1$. Then there exists a unique solution $u(x,t) \in C([0,T]; H^s(R))$ where $T > 0$ depends on $\| u_0 \|_{H^s(R)}$. If $s \geq 2$, the solution of problem (4) exists for all $T > 0$.

Proof. Assuming $D = (1 - \partial_x^2 + \varepsilon \partial_x^4)^{-1}$, we know that $D : H^s \to H^{s+4}$ is a bounded linear operator. Applying the operator D to both sides of the first equation of system (4) and then integrating the resultant equation with respect to t lead to

$$u(x,t) = u_0(x) + \int_0^t D[\partial_x(-2ku - \frac{a}{m+1}u^{m+1}) + \frac{1}{n+1}\partial_x^3(u^{n+1}) - \frac{n}{2}\partial_x(u^{n-1}u_x^2)]dt. \tag{5}$$

Suppose that both u and v are in the closed ball $B_{M_0}(0)$ of radius M_0 about the zero function in $C([0,T]; H^s(R))$ and A is the operator in the right-hand side of (5). For fixed $t \in [0,T]$,

$$\begin{aligned}
&\| \int_0^t D[\partial_x(-2ku - \frac{a}{m+1}u^{m+1}) + \frac{1}{n+1}\partial_x^3(u^{n+1}) - \frac{n}{2}\partial_x(u^{n-1}u_x^2)]dt \\
&\quad - \int_0^t D[\partial_x(-2kv - \frac{a}{m+1}v^{m+1}) + \frac{1}{n+1}\partial_x^3(v^{n+1}) - \frac{n}{2}\partial_x(v^{n-1}v_x^2)]dt \|_{H^s} \\
&\quad \leq TC_1\Big(2k \sup_{0\leq t\leq T} \| u - v \|_{H^s} + \sup_{0\leq t\leq T} \| u^{m+1} - v^{m+1} \|_{H^s} \\
&+ \sup_{0\leq t\leq T} \| u^{n+1} - v^{n+1} \|_{H^s} + \sup_{0\leq t\leq T} \| D\partial_x\big(\partial_x(u^n)\partial_x u - \partial_x(v^n)\partial_x v\big) \|_{H^s} \Big),
\end{aligned} \tag{6}$$

where C_1 may be dependent of ε. The algebraic property of $H^s(R)$ with $s > \frac{1}{2}$ derives the following

$$\begin{aligned}
\| u^{m+1} - v^{m+1} \|_{H^s} &= \| (u-v)(u^m + u^{m-1}v + \cdots + uv^{m-1} + v^m) \|_{H^s} \\
&\leq M_0^m \| (u-v) \|_{H^s}
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
&\| D\partial_x\big(\partial_x(u^n)\partial_x u - \partial_x(v^n)\partial_x v\big) \|_{H^s} \\
&\leq \| D\partial_x\big(\partial_x(u^n)\partial_x(u-v)\big) \|_{H^s} + \| D\partial_x\big(\partial_x(u^n - v^n)\partial_x v\big) \|_{H^s} \\
&\leq C_2\Big(\| \partial_x(u^n)(u_x - v_x) \|_{H^{s-2}} + \| \partial_x(u^n - v^n)\partial_x v \|_{H^{s-2}} \Big) \\
&\leq C_2 \| u^n \|_{H^s} \| u - v \|_{H^s} + \| u^n - v^n \|_{H^s} \| v \|_{H^s}, \\
&\leq C_2 M_0^n \| (u-v) \|_{H^s},
\end{aligned} \tag{8}$$

in which $s \geq 1$ is used and C_2 may depend on ε. From (5)-(8), we obtain

$$\| Au - Av \|_{H^s} \leq \theta \| u - v \|_{H^s}, \tag{9}$$

where $\theta = TC_3[2k + M_0^m + M_0^n]$ and C_3 is independent of t. Choosing T sufficiently small so that $\theta < 1$, we know that A is a contraction. Applying the above inequality yields

$$\| Au \|_{H^s} \leq \| u_0 \|_{H^s} + \theta \| u \|_{H^s} . \tag{10}$$

Choosing T sufficiently small so that $\theta M_0 + \| u_0 \|_{H^s} < M_0$, we deduce that A maps $B_{M_0}(0)$ to itself. It follows from the contraction-mapping principle that the mapping A has a unique fixed point u in $B_{M_0}(0)$.

For $s \geq 2$, using the first equation of system (4) derives

$$\frac{d}{dt} \int_R (u^2 + u_x^2 + \varepsilon u_{xx}^2) dx = 0, \tag{11}$$

from which we have the conservation law

$$\int_R (u^2 + u_x^2 + \varepsilon u_{xx}^2) dx = \int_R (u_0^2 + u_{0x}^2 + \varepsilon u_{0xx}^2) dx. \tag{12}$$

The global existence result follows from the integral form (5) and (12). ■

3 Regularity estimates of solutions for system (4)

In this section we will give the regularity estimates of solutions to the regularized initial value problem (4). Firstly, we cite two lemmas presented in [8].

Lemma 3.1(Kato and Ponce). If $r > 0$, then $H^r \bigcap L^\infty$ is an algebra. Moreover

$$\| uv \|_r \leq c\big(\| u \|_{L^\infty} \| v \|_r + \| u \|_r \| v \|_{L^\infty} \big),$$

where c is a constant depending only on r.

Lemma 3.2(Kato and Ponce). Let $r > 0$. If $u \in H^r \bigcap W^{1,\infty}$ and $v \in H^{r-1} \bigcap L^\infty$, then

$$\| [\Lambda^r, u] v \|_{L^2} \leq c\big(\| \partial_x u \|_{L^\infty} \| \Lambda^{r-1} v \|_{L^2} + \| \Lambda^r u \|_{L^2} \| v \|_{L^\infty} \big).$$

Lemma 3.3 Let $s \geq 4$ and the function $u(x,t)$ is a solution of problem (4) and the initial data $u_0(x) \in H^s$. Then the following inequalities hold

$$\| u \|_{H^1}^2 \leq c \int_R (u^2 + u_x^2 + \varepsilon u_{xx}^2) dx = c \int_R (u_0^2 + u_{0x}^2 + \varepsilon u_{0xx}^2) dx. \tag{13}$$

For $q \in (0, s-1]$, there is a constant c_1, depending only on m, n and q, such that

$$\int_R (\Lambda^{q+1}u)^2 dx \leq \int_R [(\Lambda^{q+1}u_0)^2) + \varepsilon(\Lambda^q u_{0xx})^2] dx +$$
$$c_1 \int_0^t \| u_x \|_{L^\infty} \Big(\| u \|_{H^q}^2 \big(\| u \|_{L^\infty}^{m-1} + \| u \|_{L^\infty}^{n-1} \big) + \| u \|_{H^{q+1}}^2 \| u \|_{L^\infty}^{n-1} \Big) d\tau. \quad (14)$$

For $q \in [0, s-1]$, there is a constant c_3, which is independent of ε, such that

$$(1-2\varepsilon) \| u_t \|_{H^q} \leq c \| u \|_{H^{q+1}} \Big(1 + \big(\| u \|_{L^\infty}^{m-1} + \| u \|_{L^\infty}^{n-1} \big) \| u \|_{H^1} \Big). \quad (15)$$

Proof. Using $\| u \|_{H^1}^2 \leq c \int_R (u^2 + u_x^2) dx$ and (12) derives (13).

For $q \in (0, s-1]$, applying $(\Lambda^q u)\Lambda^q$ to both sides of the first equation of system (4) and integrating with respect to x by parts, we have the identity

$$\frac{1}{2}\frac{d}{dt} \int_R ((\Lambda^q u)^2 + (\Lambda^q u_x)^2 + \varepsilon(\Lambda^q u_{xx})^2) dx = -a \int_R (\Lambda^q u)\Lambda^q(u^m u_x) dx$$
$$- \int_R (\Lambda^{q+1}u)\Lambda^{q+1}(u^n u_x) dx + \frac{n}{2} \int_R (\Lambda^q u_x)\Lambda^q(u^{n-1}u_x^2) dx + \int_R (\Lambda^q u)\Lambda^q(u^n u_x) dx. \quad (16)$$

We will estimate the terms on the right-hand side of (16) separately. For the first term, by using the Cauchy-Schwartz inequality and the Lemmas 3.1 and 3.2, we have

$$\int_R (\Lambda^q u)\Lambda^q(u^m u_x) dx = \int_R (\Lambda^q u)[\Lambda^q(u^m u_x) - u^m \Lambda^q u_x] dx + \int_R (\Lambda^q u) u^m \Lambda^q u_x dx$$
$$\leq c \| u \|_{H^q} \Big(m \| u \|_{L^\infty}^{m-1} \| u_x \|_{L^\infty} \| u \|_{H^q} + \| u_x \|_{L^\infty} \| u \|_{L^\infty}^{m-1} \| u \|_{H^q} \Big)$$
$$+ \frac{m}{2} \| u \|_{L^\infty}^{m-1} \| u_x \|_{L^\infty} \| \Lambda^q u \|_{L^2}^2$$
$$\leq c \| u \|_{H^q}^2 \| u \|_{L^\infty}^{m-1} \| u_x \|_{L^\infty} . \quad (17)$$

Using the above estimate to the second term yields

$$\int_R (\Lambda^{q+1}u)\Lambda^{q+1}(u^n u_x) dx \leq c \| u \|_{H^{q+1}}^2 \| u \|_{L^\infty}^{n-1} \| u_x \|_{L^\infty} . \quad (18)$$

For the third term, using Cauchy -Schwartz inequality and Lemma 3.1, we obtain

$$\int_R (\Lambda^q u_x)\Lambda^q(u^{n-1}u_x^2) dx \leq \| \Lambda^q u_x \|_{L^2} \| \Lambda^q(u^{n-1}u_x^2) \|_{L^2}$$
$$\leq c \| u \|_{H^{q+1}} \Big(\| u^{n-1}u_x \|_{L^\infty} \| u_x \|_{H^q} + \| u_x \|_{L^\infty} \| u^{n-1}u_x \|_{H^q} \Big)$$
$$\leq c \| u \|_{H^{q+1}}^2 \| u_x \|_{L^\infty} \| u \|_{L^\infty}^{n-1}, \quad (19)$$

in which we use $\| u^{n-1}u_x \|_{H^q} \leq c \| (u^n)_x \|_{H^q} \leq cn \| u \|_{L^\infty}^{n-1} \| u \|_{H^{q+1}}$ derived from Lemma 3.1. It follows from (16)-(19) that there exists a constant c depending only on m, n and s such that

$$\frac{1}{2}\frac{d}{dt}\int_R \big[(\Lambda^q u)^2 + (\Lambda^q u_x)^2 + \varepsilon(\Lambda^q u_{xx})^2\big]dx$$
$$\leq c \| u_x \|_{L^\infty} \Big(\| u \|_{H^q}^2 \big(\| u \|_{L^\infty}^{m-1} + \| u \|_{L^\infty}^{n-1} \big) + \| u \|_{H^{q+1}}^2 \| u \|_{L^\infty}^{n-1} \Big). \quad (20)$$

Integrating both sides of the above inequality with respect to t results in inequality (14).

To estimate the norm of u_t, we apply the operator $(1-\partial_x^2)^{-1}$ to both sides of the first equation of system (4) to obtain the equation

$$(1-\varepsilon)u_t - \varepsilon u_{xxt} =$$
$$(1-\partial_x^2)^{-1}\big[-\varepsilon u_t + \partial_x\big(-2ku - \frac{a}{m+1}u^{m+1} + \frac{1}{n+1}\partial_x^2(u^{n+1}) - \frac{n}{2}u^{n-1}u_x^2\big)\big] \quad (21)$$

Applying $(\Lambda^q u_t)\Lambda^q$ to both sides of equation (21) for $q \in (0, s-1]$ gives rise to

$$(1-\varepsilon)\int_R (\Lambda^q u_t)^2 dx + \varepsilon \int_R (\Lambda^q u_{xt})^2 dx$$
$$= \int_R (\Lambda^q u_t)\Lambda^{q-2}\Big[-\varepsilon u_t + \partial_x\big(-2ku - \frac{a}{m+1}u^{m+1} + \frac{1}{n+1}\partial_x^2(u^{n+1})$$
$$-\frac{n}{2}u^{n-1}u_x^2\big)\Big]d\tau. \quad (22)$$

On the right-hand of equation (22), we have

$$\int_R (\Lambda^q u_t)\Lambda^{q-2}(-\varepsilon u_t - 2ku_x)dx \leq \varepsilon \| u_t \|_{H^q}^2 + 2k \| u_t \|_{H^q} \| u \|_{H^q}, \quad (23)$$

and

$$\int_R (\Lambda^q u_t)(1-\partial_x^2)^{-1}\Lambda^q \partial_x(-\frac{a}{m+1}u^{m+1} - \frac{1}{2}u^{n-1}u_x^2)dx$$
$$\leq c \| u_t \|_{H^q} \Bigg(\int_R (1+\xi^2)^{q-1} \times$$
$$\Big[\int_R \big[-\frac{a}{m+1}\widehat{u^m}(\xi-\eta)\widehat{u}(\eta) - \frac{n}{2}\widehat{u^{n-1}u_x}(\xi-\eta)\widehat{u_x}(\eta)\big]d\eta\Big]^2\Bigg)^{\frac{1}{2}}$$
$$\leq c \| u_t \|_{H^q} \| u \|_{H^1} \| u \|_{H^{q+1}} (\| u \|_{L^\infty}^{m-1} + \| u \|_{L^\infty}^{n-1}). \quad (24)$$

Since

$$\int (\Lambda^q u_t)(1-\partial_x^2)^{-1}\Lambda^q \partial_x^2(u^n u_x) = -\int (\Lambda^q u_t)\Lambda^q(u^n u_x) + \int (\Lambda^q u_t)(1-\partial_x^2)^{-1}\Lambda^q(u^n u_x),$$
$$(25)$$

it follows from Young's inequality $\| f \star g \|_{H^r} \leq \| f \|_{H^{p_1}} \| g \|_{H^{p_2}}, \frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{r}$, the inequality $(1+\xi^2)^l \leq [(1+(\xi-\eta)^2)^l + (1+\eta^2)^l], l > 0$, and Lemma 3.1, that

$$
\begin{aligned}
&\int (\Lambda^q u_t)\Lambda^q(u^n u_x)dx \\
&\leq c \| u_t \|_{H^q} \left(\int_R c\Big(\int_R [(1+(\xi-\eta)^2)^{\frac{q+1}{2}} + (1+\eta^2)^{\frac{q+1}{2}}] \hat{u^n}(\xi-\eta)\hat{u}(\eta)d\eta\Big)^2 d\xi \right)^{\frac{1}{2}} \\
&\leq c \| u_t \|_{H^q} \left(\| \widehat{\Lambda^{q+1} u^n} \star \widehat{u} \|_{L^2} + \| \widehat{u^n} \star \widehat{\Lambda^{q+1} u} \|_{L^2} \right) \\
&\leq c \| u_t \|_{H^q} \| u \|_{L^\infty}^{n-1} \| u \|_{H^1} \| u \|_{H^{q+1}} \qquad (26)
\end{aligned}
$$

and

$$\int (\Lambda^q u_t)(1-\partial_x^2)^{-1}\Lambda^q(u^n u_x)dx \leq c \| u_t \|_{H^q} \| u \|_{L^\infty}^{n-1} \| u \|_{H^1} \| u \|_{H^{q+1}} . \quad (27)$$

Applying (21)-(27) yields the inequality

$$(1-2\varepsilon) \| u_t \|_{H^q} \leq c \| u \|_{H^{q+1}} \Big(1 + \big(\| u \|_{L^\infty}^{m-1} + \| u \|_{L^\infty}^{n-1} \big) \| u \|_{H^1} \Big) \quad (28)$$

for some constant $c > 0$. This completes the proof. ■

For a real number $s > 0$, suppose that the function $u_0(x)$ is in $H^s(R)$, and let $u_{\varepsilon 0}$ be the convolution $u_{\varepsilon 0} = \phi_\varepsilon \star u_0$ of the function $\phi_\varepsilon(x) = \varepsilon^{-\frac{1}{4}}\phi(\varepsilon^{-\frac{1}{4}}x)$ and u_0 such that the Fourier transform $\widehat{\phi}$ of ϕ satisfies $\widehat{\phi} \in C_c^\infty$, $\widehat{\phi(\xi)} \geq 0$, and $\widehat{\phi(\xi)} = 1$ for any $\xi \in (-1,1)$. Thus we have $u_{\varepsilon 0}(x) \in C_0^\infty$. It follows from Theorem 2.1 that for each ε satisfying $0 < \varepsilon < \frac{1}{4}$, the Cauchy problem

$$\begin{cases} u_t - u_{xxt} + \varepsilon u_{xxxxt} = \partial_x(-2ku - \frac{a}{m+1}u^{m+1}) + \frac{1}{n+1}\partial_x^3(u^{n+1}) - \frac{n}{2}\partial_x(u^{n-1}u_x^2), \\ u(x,0) = u_{\varepsilon 0}(x), \end{cases} \quad (29)$$

has a unique solution $u_\varepsilon(x,t) \in C^\infty([0,\infty); H^\infty)$. To show that u_ε is convergent to a solution of problem (3), we cite the following lemma presented in [9].

Lemma 3.4 Under the above assumptions, the following estimates hold for any ε with $0 < \varepsilon < \frac{1}{4}$

$$\| u_{\varepsilon 0} \|_{H^q} \leq c, \quad \text{if} \quad q \leq s, \quad (30)$$

$$\| u_{\varepsilon 0} \|_{H^q} \leq c\varepsilon^{\frac{s-q}{4}}, \quad \text{if} \quad q > s, \quad (31)$$

where c is a constant independent of ε and $s > 0$.

From Lemmas 3.3 and 3.4, we know that there exists a positive constant c_0 independent of ε such that

$$\| u_\varepsilon \|_{L^\infty} < c_0, \qquad \| u_\varepsilon \|_{H^1} < c_0. \quad (32)$$

4 Existence of weak solution

The task of this section is to give a sufficient condition which guarantees that the existence of weak solution for the generalized Camassa-Holm equation (3) in the Sobolev space H^s with $1 < s \leq \frac{3}{2}$. Firstly, we use the regularized problem (29) to estimate norms of its solutions, showing that they are bounded. When the parameter ε is sufficiently small, it gives the weak convergence of these solutions to a solution of the Camassa-Holm equation (3).

Theorem 4.1 If $u_0(x) \in H^s(R)$ with $s \in [1, \frac{3}{2}]$ such that $\| u_{0x} \|_{L^\infty} < \infty$. Let $u_{\varepsilon 0}$ be defined as in Section 3. Then there exist constants $T > 0$ and c independent of ε such that the solution u_ε of problem (29) satisfies $\| u_{\varepsilon x} \|_{L^\infty} \leq c$.

Proof. Using notation $u = u_\varepsilon$ and differentiating with respect to x on both sides of the first equation of problem (29) give rise to

$$(1-\varepsilon)u_{xt} - \varepsilon u_{xxxt} + \frac{1}{n+1}\partial_x^2 u^{n+1} - \frac{n}{2}u^{n-1}u_x^2 = 2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+1}u^{n+1}$$
$$-\Lambda^{-2}\big[\varepsilon u_{xt} + 2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+1}u^{n+1} + \frac{n}{2}u^{n-1}u_x^2\big]. \tag{33}$$

Letting $p > 0$ be an integer and multiplying the above equation by $(u_x)^{2p+1}$ and then integrating the resulting equation with respect to x yield the equality

$$\frac{1-\varepsilon}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx - \varepsilon\int_R (u_x)^{2p+1}u_{xxxt}dx + \frac{pn}{2p+2}\int_R (u_x)^{2p+3}u^{n-1}dx$$
$$= \int_R (u_x)^{2p+1}(2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{n+1}u^{n+1})dx$$
$$- \int_R (u_x)^{2p+1}\Lambda^{-2}\big[\varepsilon u_{xt} + 2ku + \frac{a}{m+1}u^{m+1} - \frac{u^{n+1}}{n+1} + \frac{n}{2}u^{n-1}u_x^2\big]dx. \tag{34}$$

Applying the Hölder's inequality yields

$$\frac{1-\varepsilon}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx$$
$$\leq \Big\{\varepsilon\big(\int_R |u_{xxxt}|^{2p+2}dx\big)^{\frac{1}{2p+2}} + |2k|\big(\int_R |u|^{2p+2}dx\big)^{\frac{1}{2p+2}}$$
$$+\frac{a}{m+1}\big(\int_R |u^{m+1}|^{2p+2}dx\big)^{\frac{1}{2p+2}} + \frac{1}{n+1}\big(\int_R |u^{n+1}|^{2p+2}dx\big)^{\frac{1}{2p+2}}$$
$$+\big(\int_R |G|^{2p+2}dx\big)^{\frac{1}{2p+2}}\Big\}\big(\int_R |u_x|^{2p+2}dx\big)^{\frac{2p+1}{2p+2}}$$
$$+\frac{np}{2p+2} \| u_x \|_{L^\infty} \| u \|_{L^\infty}^{n-1} \int_R |u_x|^{2p+2}dx. \tag{35}$$

or

$$
\begin{aligned}
&(1-\varepsilon)\frac{d}{dt}\Big(\int_R (u_x)^{2p+2}dx\Big)^{\frac{1}{2p+2}}\\
&\le \varepsilon\Big(\int_R |u_{xxxt}|^{2p+2}dx\Big)^{\frac{1}{2p+2}} + |2k|\Big(\int_R |u|^{2p+2}dx\Big)^{\frac{1}{2p+2}}\\
&+\frac{a}{m+1}\Big(\int_R |u^{m+1}|^{2p+2}dx\Big)^{\frac{1}{2p+2}} + \frac{1}{n+1}\Big(\int_R |u^{n+1}|^{2p+2}dx\Big)^{\frac{1}{2p+2}}\\
&+\Big(\int_R |G|^{2p+2}dx\Big)^{\frac{1}{2p+2}} + \frac{np}{2p+2}\parallel u_x \parallel_{L^\infty}\parallel u \parallel_{L^\infty}^{n-1}\Big(\int_R |u_x|^{2p+2}dx\Big)^{\frac{1}{2p+1}}, \quad (36)
\end{aligned}
$$

where

$$G = \Lambda^{-2}\big[\varepsilon u_{xt} + 2ku + \frac{a}{m+1}u^{m+1} - \frac{u^{n+1}}{n+1} + \frac{n}{2}u^{n-1}u_x^2\big].$$

Since $\parallel f \parallel_{L^p} \to \parallel f \parallel_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \bigcap L^2$, integrating with respect to t and taking the limit as $p \to \infty$ on both sides of inequality (36) result in the estimate

$$
\begin{aligned}
&(1-\varepsilon)\parallel u_x \parallel_{L^\infty} \le (1-\varepsilon)\parallel u_{0x} \parallel_{L^\infty}\\
&+\int_R \Big[\varepsilon \parallel u_{xxxt} \parallel_{L^\infty} + c\big(\parallel u \parallel_{L^\infty} + \parallel u^{m+1} \parallel_{L^\infty} + \parallel u^{n+1} \parallel_{L^\infty} + \parallel G \parallel_{L^\infty}\big)\\
&+\frac{n}{2}\parallel u \parallel_{L^\infty}^{n-1}\parallel u_x \parallel_{L^\infty}^2\Big]d\tau. \quad (37)
\end{aligned}
$$

Using the algebraic property of $H^s(R)$ with $s > \frac{1}{2}$ and the inequality (32) yields

$$\parallel u^{m+1} \parallel_{L^\infty} \le c \parallel u^{m+1} \parallel_{H^{\frac{1}{2}+}} \le c \parallel u^{m+1} \parallel_{H^1} \le c \parallel u \parallel_{H^1}^{m+1} \le c_1, \quad (38)$$

and

$$
\begin{aligned}
&\parallel G \parallel_{L^\infty} \le c \parallel G \parallel_{H^{\frac{1}{2}+}}\\
&= c \parallel \Lambda^{-2}\big[\varepsilon u_{xt} + 2ku + \frac{a}{m+1}u^{m+1} - \frac{u^{n+1}}{n+1} + \frac{n}{2}u^{n-1}u_x^2\big] \parallel_{H^{\frac{1}{2}+}}\\
&\le c\big(\parallel \Lambda^{-2}u_{xt} \parallel_{H^{\frac{1}{2}+}} + \parallel \Lambda^{-2}\big(u^{n-1}u_x^2\big) \parallel_{H^{\frac{1}{2}+}}\big) + c_2\\
&\le c(\parallel u_t \parallel_{L^2} + \parallel u^{n-1}u_x^2 \parallel_{H^0}) + c_2\\
&\le c_3(\parallel u_t \parallel_{L^2} + \parallel u_x \parallel_{L^\infty}\parallel u \parallel_{H^1}) + c_2\\
&\le c_3(\parallel u_t \parallel_{L^2} + \parallel u_x \parallel_{L^\infty}^2) + c_2, \quad (39)
\end{aligned}
$$

where c, c_1, c_2 and c_3 are constants and are independent of ε. Using (15), (38) and (39), we have

$$\parallel G \parallel_{L^\infty} \le c_4(\parallel u_x \parallel_{L^\infty}^2) + 1), \quad (40)$$

where c_4 is a constant independent of ε. Moreover, for any fixed $r \in (\frac{1}{2}, 1)$, there exists a constant c_r such that $\| u_{xxxt} \|_{L^\infty} \leq c_r \| u_{xxxt} \|_{H^r} \leq c_r \| u_t \|_{H^{r+3}}$. Using (15) and (38) yields

$$\| u_{xxxt} \|_{L^\infty} \leq c \| u \|_{H^{r+4}} . \tag{41}$$

Making use of the Gronwall's inequality to (14) with $q = r+3$ and $u = u_\varepsilon$, we have

$$\| u \|^2_{H^{r+4}} \leq \Big(\int_R (\Lambda^{r+4} u_0)^2 + \varepsilon(\Lambda^{r+3} u_{0xx})^2 \Big) \exp \Big(c \int_0^t \| u_x \|_{L^\infty} d\tau \Big). \tag{42}$$

From (31) and (42), one has

$$\| u_{xxxt} \|_{L^\infty} \leq c \varepsilon^{\frac{s-r-4}{4}} \exp \Big(c \int_0^t \| u_x \|_{L^\infty} d\tau \Big). \tag{43}$$

For $\varepsilon < \frac{1}{4}$, it follows from (37),(38), (40) and (43) that

$$\| u_x \|_{L^\infty} \leq \| u_{0x} \|_{L^\infty} + c \int_0^t [\varepsilon^{\frac{s-r}{4}} \exp \Big(c \int_0^\tau \| u_x \|_{L^\infty} d\varsigma \Big) + \frac{1}{2} \| u_x \|^2_{L^\infty} + 1 \Big] d\tau. \tag{44}$$

Using the Theorem presented at page 51 in [6] and combining (44), we know that there are constants $T > 0$ and $c > 0$ independent of ε such that $\| u_x \|_{L^\infty} \leq c$. ■

A direct use of Theorem 5.1 results in the following theorem which states the existence of a weak solution to equation (3).

Theorem 4.2 Suppose that $u_0(x) \in H^s$ with $1 < s \leq \frac{3}{2}$ and $\| u_{0x} \|_{L^\infty} < \infty$. Then there exists a $T > 0$ such that equation (3) associated with initial value $u_0(x)$ has a solution $u(x,t) \in L^2([0,T], H^s)$ in the sense of distribution and $u_x \in L^\infty([0,T] \times R)$.

Proof. From Theorem 4.1, we know that $\{u_{\varepsilon_n x}\} (\varepsilon_n \to 0)$ is bounded in the space L^∞. Thus, the sequences $\{u^2_{\varepsilon_n}\}$ and $\{u^2_{\varepsilon_n x}\}$ are weakly convergent to u^2 and u_x^2 in $L^2[0,T], H^r(-R,R)$ for any $r \in [0, s-1)$, respectively. Considering equation (3) and choosing any $f \in C_0^\infty$, we have

$$-\int_0^T \int_R u(f_t - f_{xxt}) dx dt = \int_0^T \int_R \Big[\Big(2ku + \frac{a}{m+1} u^{m+1} + \frac{n}{2} (u^{n-1} u_x^2) \Big) f_x - \frac{1}{n+1} u^{n+1} f_{xxx} \Big] dx dt, \tag{45}$$

with $u(x,0) = u_0(x)$. Since $X = L^1([0,T] \times R)$ is a separable Banach space and $\{u_{\varepsilon_n x}\}$ is a bounded sequence in the dual space $X^* = L^\infty([0,T] \times R)$ of X, there exists a subsequence of $\{u_{\varepsilon_n x}\}$, still denoted by $\{u_{\varepsilon_n x}\}$, weakly star convergent to a function v in $L^\infty([0,T] \times R)$. It derives from the $\{u_{\varepsilon_n x}\}$ weakly convergent to u_x in $L^2([0,T] \times R)$ that $u_x = v$ almost everywhere. Thus, we obtain $u_x \in L^\infty([0,T] \times R)$. ■

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