International Mathematical Forum, 5, 2010, no. 41, 2001 - 2009

Exact Solutions to a Generalized BBM Equation with Variable Coefficients[1]

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Abstract

An auxiliary equation technique is applied to investigate a generalized Benjamin-Bona-Mahony equation with variable coefficients. Many exact traveling wave solutions are obtained which include algebraic solutions, solitons, solitary wave solutions and trigonometric solutions.

Mathematics Subject Classification: 35Q53, 35B35

Keywords: BBM equation; the auxiliary equation method; Symbolic computation system; solitons; trigonometric solutions

1 Introduction

Exact solutions of some partial differential equations are of fundamental importance in science because the phenomena and dynamical process modeled by the equations can be better understood by analyzing the properties of the solutions. Huang and Zhang [1] used the extended honmogeneous balance method and got many new exact solutions of the generalized Kadomtsev-Petviashvili equation with variable coefficients. By decomposing the time and space variables of nonlinear partial differential equations into two integrable ordinary differential equation, Ma and Wu [2] have found some exact solutions of KdV, mKdV and KPP equations. Hua and Li [3] extended previous development approximate method and investigated the cubic-quintic nonlinear Schrödinger

[1]This work is supported by the key project of Chinese Ministry of Education (109140).
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equation, all of the simplified solitary wave solutions, kinks and anti-kinks about the equation are obtained. The auxiliary equation method is a powerful mathematical techniques to obtain exact solutions of many nonlinear equations, Sirendaoreji [4, 5] developed it to solve the Klein-Gordon, the sine-Gordon equation and obtained many new solutions. Other approaches such as the inverse scattering method, the B*ä*cklund transformation, the Darboux transformation, the painlev*é* analysis, the tri-Hamiltonian operators, the finite difference method, the Adomian decomposition method, the sin-cos antaze method, the Jacobi elliptic function expansion methods, the hyperbolic tangent method, the mapping method, the reader is referred to [4–6, 8] and the references therein.

Benjamin, Bona and Mahony [7] established the model

$$u_t + au_x - bu_{xxt} + k(u^2)_x = 0, \tag{1}$$

which was called BBM equation. It is used as an alternative to the KdV equation which describes unidirectional propagation of weakly long dispersive waves [7].

In the present work, we study another generalized BBM equation which is written by

$$u_t + a(t)u_x - b(t)(u^n)_{xxt} + k(t)(u^n)_x = 0, \tag{2}$$

where $n > 0$, $a(t) \neq 0, b(t) \neq 0$ and $k(t) \neq 0$ are functions of t. By making use of the auxiliary equation method with the help of symbolic computation system, we shall focus on deriving the exact travelling wave solutions including algebraic solutions, solitons and trigonometric function solutions for Eq.(2).

2 Brief description of the method

For the nonlinear partial differential equation

$$P(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt} \cdots) = 0, \tag{3}$$

we make the transformation $\xi = p(t)x + q(t)$ and assume that the solutions of Eq.(3) satisfy the ansatz

$$u(x,t) = g_0(t) + g_1(t)z(\xi), \tag{4}$$

where $g_0(t), g_1(t), p(t)$ and $q(t)$ are all unknown functions of t, and $z(\xi)$ satisfies the auxiliary ordinary differential equation

$$(\frac{dz}{d\xi})^2 = a_1 + a_2 z + a_3 z^2, \tag{5}$$

where $a_i (i = 1, 2, 3)$ are real constants. Jeffrey [9] obtained the solutions of Eq.(5), which are written in the Table.

Table 1 Solutions of Eq.(5)
$\varepsilon = \pm 1$

No	*Condition*	The expressions of $z(\xi)$
1	$a_1 \neq 0,\ a_2 = 0,\ a_3 < 0$	$z(\xi) = \varepsilon\sqrt{-\frac{a_1}{a_3}}\sin(\sqrt{-a_3}\xi)$
		$z(\xi) = \varepsilon\sqrt{-\frac{a_1}{a_3}}\cos(\sqrt{-a_3}\xi)$
2	$a_1 \neq 0,\ a_2 = 0,\ a_3 > 0$	$z(\xi) = \varepsilon\sqrt{\frac{a_1}{a_3}}\sinh(\sqrt{a_3}\xi)$
		$z(\xi) = i\varepsilon\sqrt{\frac{a_1}{a_3}}\cosh(\sqrt{a_3}\xi)$
3	$a_1 \neq 0,\ a_2 = 0,\ a_3 \neq 0$	$z(\xi) = \frac{e^{2\sqrt{a_3}\xi} - a_1}{2\sqrt{a_3}e^{\sqrt{a_3}\xi}}$
4	$a_3 > 0,\ a_2^2 - 4a_1a_3 \neq 0$	$z(\xi) = \frac{i\varepsilon\sqrt{a_2^2-4a_1a_3}}{2a_3}\sinh(\sqrt{a_3}\xi) - \frac{a_2}{2a_3}$
		$z(\xi) = \frac{\varepsilon\sqrt{a_2^2-4a_1a_3}}{2a_3}\cosh(\sqrt{a_3}\xi) - \frac{a_2}{2a_3}$
5	$a_3 < 0,\ a_2^2 - 4a_1a_3 \neq 0$	$z(\xi) = \frac{\varepsilon\sqrt{a_2^2-4a_1a_3}}{2a_3}\sin(\sqrt{-a_3}\xi) - \frac{a_2}{2a_3}$
		$z(\xi) = \frac{\varepsilon\sqrt{a_2^2-4a_1a_3}}{2a_3}\cos(\sqrt{-a_3}\xi) - \frac{a_2}{2a_3}$
6	$a_3 \neq 0,\ a_2^2 - 4a_1a_3 = 0$	$z(\xi) = \frac{1}{2a_3}e^{\sqrt{a_3}\xi} - \frac{a_2}{2a_3}$

Substituting Eqs.(4) and (5) into Eq.(3) and letting the coefficients of all powers of $z^i(\xi)(i = 0, 1, 2 \cdots)$ and $x^k z^j \sqrt{a_1 + a_2 z + a_3 z^2}(k = 0, 1, j = 0, 1, 2 \cdots)$ to be zero in the resulting equation yeild several over-determined PDEs with respect to unknown functions $g_i(t)(i = 0, 1), p(t)$ and $q(t)$. Solving these equations with the help of the symbolic computation system Matlab, we can get the exact expressions of $g_i(t)(i = 0, 1)$ and ξ. Thus, according to Eq.(4) and the solutions of the auxiliary equation, we can obtain exact solutions of

Eq.(3).

3 Exact traveling wave solutions for the Eq.(2)

First of all, we give a transformation of $u^{n-1}(x,t) = w(x,t)$ to Eq.(2), which yields

$$\begin{cases} u_t = \frac{1}{n-1} w^{\frac{1}{n-1}}, \\ u_x = \frac{1}{n-1} w^{\frac{1}{n-1}}, \\ (u^n)_x = \frac{n}{n-1} w^{\frac{1}{n-1}} w_x, \\ (u^n)_{xxt} = \frac{n w^{\frac{1}{n-1}-2}}{(n-1)^2} \Big[\frac{2-n}{n-1} (w_x)^2 w_t + 2w w_x w_{xt} + w w_t w_{xx} + (n-1) w^2 w_{xxt} \Big]. \end{cases} \tag{6}$$

Substituting Eq.(6) into (2), we obtain

$$\begin{aligned} & w w_t + a(t) w w_x - \frac{b(t)n(2-n)}{(n-1)^2} w_t (w_x)^2 - \frac{2b(t)n}{n-1} w w_x w_{xt} \frac{b(t)n}{n-1} w w_t w_{xx} \\ & - b(t) n w^2 w_{xxt} + k(t) n w^2 w_x = 0. \end{aligned} \tag{7}$$

Assuming that the solution of Eq.(7) has the following form

$$w(x,t) = f(t) + h(t) z(\xi), \quad \xi = p(t)x + q(t), \tag{8}$$

where $f(t), h(t), p(t), q(t)$ are functions of t to be determined later and $z(\xi)$ satisfies Eq.(5).

Substituting Eqs.(5) and (8) into Eq.(7) and letting each coefficient of

$x^j z^i (0 \leq i \leq 6, j = 0, 1)$ and $\sqrt{a_1 + a_2 z + a_3 z^2}$ to be zero, we obtain

$$
\begin{aligned}
& p'(t) = 0, \qquad (9) \\
& \frac{-2b(t)nh^3(t)p^2(t)a_3q'(t)}{(n-1)^2} - \frac{3b(t)nh^3(t)p^2(t)a_3q'(t)}{n-1} + k(t)nh^3(t)p(t) \\
& + \frac{n^2b(t)h^3(t)p^2(t)q'(t)a_3}{(n-1)^2} - b(t)nh^3(t)p^2(t)a_3q'(t) = 0, \qquad (10) \\
& 2k(t)nh^2(t)p(t)f(t) - \frac{2b(t)nh^3(t)p^2(t)q'(t)a_2}{(n-1)^2} - \frac{3b(t)nh^3(t)p^2(t)q'(t)a_2}{2(n-1)} \\
& + \frac{n^2b(t)h^3(t)p^2(t)q'(t)a_2}{(n-1)^2} + h^2(t)q'(t) - 2b(t)nh^2(t)p^2(t)a_3f(t)q'(t) \\
& - \frac{3b(t)nh^2p^2a_3f(t)q_t}{n-1} + a(t)h^2(t)p(t) = 0, \qquad (11) \\
& a(t)h(t)p(t)f(t) + \frac{n^2b(t)h^3(t)p^2(t)q'(t)a_1}{(n-1)^2} - \frac{3b(t)nh^2(t)p^2(t)f(t)a_2q'(t)}{2(n-1)} \\
& - \frac{2b(t)nh^3(t)p^2(t)q'(t)a_1}{(n-1)^2} + k(t)nh(t)p(t)f^2(t) + h(t)q'(t)f(t) \\
& - b(t)nh(t)p^2(t)a_3f^2(t)q'(t) = 0. \qquad (12)
\end{aligned}
$$

Solving the above equations by the use of Matlab, we acquire

$$
\left\{
\begin{array}{l}
f(t) = \frac{(a_2^2 - 4a_1a_3 \pm \sqrt{a_2^4 - 4a_1a_2^2a_3})[p^2n^2a_3a(t)b(t) + k(t)(n-1)^2]}{b(t)k(t)np^2(n+1)(4a_1a_3^2 - a_2^2a_3)}, \\
h(t) = \mp \frac{2[n^2a_3p^2a(t)b(t) + k(t)(n-1)^2]}{k(t)b(t)n(n+1)p^2\sqrt{a_2^2 - 4a_1a_3}}, \\
q(t) = \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2} dt,
\end{array}
\right. \qquad (13)
$$

where $a_1, a_2, a_3 \neq 0$ and $p \neq 0$ are arbitrary constants.

From Eqs.(8), (13) and the solutions of $z(\xi)$ listed in Table 1, we obtain the exact solutions of $w(x, t)$. According to the transformation $u^{n-1}(x, t) = w$, we acquire the following exact solutions of Eq.(2)

$$u_1(x,t) = \Big\{(a(t)p^2b(t)a_3n^2 + k(t)(n-1)^2)\Big[\frac{-1}{b(t)k(t)np^2(n+1)a_3} + \frac{\varepsilon}{k(t)b(t)n(n+1)p^2a_3} sin(\sqrt{-a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2}dt))\Big]\Big\}^{\frac{1}{n-1}},$$
$$(a_1 \neq 0, a_3 < 0), \quad (14)$$

$$u_2(x,t) = \Big\{(a(t)p^2b(t)a_3n^2 + k(t)(n-1)^2)\Big[\frac{-1}{b(t)k(t)np^2(n+1)a_3} + \frac{\varepsilon}{k(t)b(t)n(n+1)p^2a_3} cos(\sqrt{-a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2}dt))\Big]\Big\}^{\frac{1}{n-1}},$$
$$(a_1 \neq 0, a_3 < 0), \quad (15)$$

$$u_3(x,t) = \Big\{(a(t)p^2b(t)a_3n^2 + k(t)(n-1)^2)\Big[\frac{-1}{b(t)k(t)np^2(n+1)a_3} + \frac{i\varepsilon}{k(t)b(t)n(n+1)p^2a_3} sinh(\sqrt{a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2}dt))\Big]\Big\}^{\frac{1}{n-1}},$$
$$(a_1 \neq 0, a_3 > 0), \quad (16)$$

$$u_4(x,t) = \Big\{(a(t)p^2b(t)a_3n^2 + k(t)(n-1)^2)\Big[\frac{-1}{b(t)k(t)np^2(n+1)a_3} + \frac{\varepsilon}{k(t)b(t)n(n+1)p^2a_3} cosh(\sqrt{a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2}dt))\Big]\Big\}^{\frac{1}{n-1}},$$
$$(a_1 \neq 0, a_3 > 0), \quad (17)$$

$$u_5(x,t) = \Big\{(p^2a_3n^2a(t)b(t) + k(t)(n-1)^2)\Big[\frac{-1}{b(t)k(t)np^2a_3(n+1)} + \frac{\varepsilon(e^{2\sqrt{a_3}(px+\int \frac{k(t)(n-1)^2}{b(t)pa_3n^2}dt)} - a_1)}{2k(t)b(t)n(n+1)p^2a_3\sqrt{-a_1}e^{\sqrt{a_3(px+\int \frac{k(n-1)^2}{b(t)pa_3n^2}dt)}}}\Big]\Big\}^{\frac{1}{n-1}},$$
$$(a_1 \neq 0, a_3 \neq 0), \quad (18)$$

$$u_6(x,t) = \Bigg\{ \Big[a(t)p^2b(t)a_3n^2 + k(t)(n-1)^2\Big] \times \Big[\frac{-a_2^2 + 4a_1a_3 + 2\varepsilon a_2\sqrt{a_2^2 - 4a_1a_3}}{b(t)k(t)np^2(n+1)(a_2^2 - 4a_1a_3)a_3} + \frac{i\varepsilon}{k(t)b(t)n(n+1)p^2a_3} \cdot sinh(\sqrt{a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2}dt))\Big]\Bigg\}^{\frac{1}{n-1}},$$
$$(a_2 < 0, a_3 > 0, a_2^2 - 4a_1a_3 \neq 0), \quad (19)$$

$$u_7(x,t) = \Bigg\{ \Big[a(t)p^2b(t)a_3n^2 + k(t)(n-1)^2\Big] \times \Big[\frac{-a_2^2 + 4a_1a_3 + 2\varepsilon a_2\sqrt{a_2^2 - 4a_1a_3})}{b(t)k(t)np^2(n+1)(a_2^2 - 4a_1a_3)a_3} - \frac{i\varepsilon}{k(t)b(t)n(n+1)p^2a_3} \cdot sinh(\sqrt{a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2}dt))\Big]\Bigg\}^{\frac{1}{n-1}},$$
$$(a_2 < 0, a_3 > 0, a_2^2 - 4a_1a_3 \neq 0), \quad (20)$$

$$u_8(x,t) = \Bigg\{ \Big[a(t)p^2b(t)a_3n^2 + k(t)(n-1)^2\Big] \times \Big[\frac{-a_2^2 + 4a_1a_3 + 2\varepsilon a_2\sqrt{a_2^2 - 4a_1a_3})}{b(t)k(t)np^2(n+1)(a_2^2 - 4a_1a_3)a_3} + \frac{\varepsilon}{k(t)b(t)n(n+1)p^2a_3} \cdot cosh(\sqrt{a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2}dt))\Big]\Bigg\}^{\frac{1}{n-1}},$$
$$(a_2 < 0, a_3 > 0, a_2^2 - 4a_1a_3 \neq 0), \quad (21)$$

$$u_9(x,t) = \Bigg\{ \Big[a(t)p^2b(t)a_3n^2 + k(t)(n-1)^2\Big] \times \Big[\frac{-a_2^2 + 4a_1a_3 + 2\varepsilon a_2\sqrt{a_2^2 - 4a_1a_3})}{b(t)k(t)np^2(n+1)(a_2^2 - 4a_1a_3)a_3} - \frac{\varepsilon}{k(t)b(t)n(n+1)p^2a_3} \cdot cosh(\sqrt{a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2}dt))\Big]\Bigg\}^{\frac{1}{n-1}},$$
$$(a_2 < 0, a_3 > 0, a_2^2 - 4a_1a_3 \neq 0), \quad (22)$$

$$u_{10}(x,t) = \Bigg\{ \Big[a(t)p^2b(t)a_3n^2 + k(t)(n-1)^2\Big] \times \Big[\frac{-a_2^2 + 4a_1a_3 + 2\varepsilon a_2\sqrt{a_2^2 - 4a_1a_3})}{b(t)k(t)np^2(n+1)(a_2^2 - 4a_1a_3)a_3} - \frac{\varepsilon}{k(t)b(t)n(n+1)p^2a_3} \cdot sin(\sqrt{-a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2}dt))\Big]\Bigg\}^{\frac{1}{n-1}},$$
$$(a_2 < 0, a_3 < 0, a_2^2 - 4a_1a_3 \neq 0), \quad (23)$$

$$u_{11}(x,t) = \Bigg\{ \Big[a(t)p^2 b(t)a_3 n^2 + k(t)(n-1)^2\Big] \times \Big[\frac{-a_2^2 + 4a_1a_3 + 2\varepsilon a_2\sqrt{a_2^2 - 4a_1a_3})}{b(t)k(t)np^2(n+1)(a_2^2 - 4a_1a_3)a_3} + \frac{\varepsilon}{k(t)b(t)n(n+1)p^2a_3} \cdot sin(\sqrt{-a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2} dt))\Big]\Bigg\}^{\frac{1}{n-1}},$$
$$(a_2 < 0, a_3 < 0, a_2^2 - 4a_1a_3 \neq 0), \qquad (24)$$

$$u_{12}(x,t) = \Bigg\{ \Big[a(t)p^2 b(t)a_3 n^2 + k(t)(n-1)^2\Big] \times \Big[\frac{-a_2^2 + 4a_1a_3 + 2\varepsilon a_2\sqrt{a_2^2 - 4a_1a_3})}{b(t)k(t)np^2(n+1)(a_2^2 - 4a_1a_3)a_3} - \frac{\varepsilon}{k(t)b(t)n(n+1)p^2a_3} \cdot cos(\sqrt{-a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2} dt))\Big]\Bigg\}^{\frac{1}{n-1}},$$
$$(a_2 < 0, a_3 < 0, a_2^2 - 4a_1a_3 \neq 0), \qquad (25)$$

$$u_{13}(x,t) = \Bigg\{ \Big[a(t)p^2 b(t)a_3 n^2 + k(t)(n-1)^2\Big] \times \Big[\frac{-a_2^2 + 4a_1a_3 + 2\varepsilon a_2\sqrt{a_2^2 - 4a_1a_3})}{b(t)k(t)np^2(n+1)(a_2^2 - 4a_1a_3)a_3} + \frac{\varepsilon}{k(t)b(t)n(n+1)p^2a_3} \cdot cos(\sqrt{-a_3}(px + \int \frac{k(t)(n-1)^2}{b(t)pa_3n^2} dt))\Big]\Bigg\}^{\frac{1}{n-1}},$$
$$(a_2 < 0, a_3 < 0, a_2^2 - 4a_1a_3 \neq 0). \qquad (26)$$

4 Conclusion

In this paper, by using the auxiliary equation method with the help of the symbolic computation system Matlab, we study a generalized Benjamin-Bona-Mahony partial differential equation with variable coefficients. Under different circumstances, many exact solutions are obtained which include algebraic solutions, soliton wave solutions and triangular function solutions. Furthermore, it is worthwhile mention that the ansatz method can also be applied to many other evolution equations.

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Received: January, 2010