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Observer for Perturbed Linear Discrete Systems[1]

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Abstract

The focus of this paper is on the conception of observers for a class of finite dimensional systems in which the dynamics are partially unknown. Here, we start by considering the class of the dynamics with a single unknown element. Then we move to studying the case of several unknown elements. We give sufficient conditions in which the existence of an observer is assured. We illustrate the obtained results by some examples and numerical simulations.

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1 Introduction

The observer design problem for linear finite dimensional systems, has been the concern of many researchers, with the objective to estimate the state of the system with a completely known dynamic matrix and an unknown initial state. Important results on the observer design problem in this case can be found in the literature(see [2], [3], [5], [6], [12], [13], [16], [17], [19], [20] and references therein). In practice, The exact and full knowledge of the dynamics of the system is generally impossible. Then it is very important to take into account the uncertainties that affect the dynamics of the system. In this work, the emphasis will be on the observer design problem, for linear finite dimensional systems, with unknown dynamics. We are going to study four cases according to the position of the unknown elements of the dynamics. In case one, there will be only one unknown element of the dynamics. In the second case, one whole column of the dynamics will be unknown. In the third case, the diagonal elements of the dynamics will be unknown. Finally, in the fourth case, there will be several unknown elements of the dynamics. In all cases we give a sufficient and necessary condition on the known part of the dynamics, in which our objective is achievable.

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2 Problem formulation

Let us consider the linear discrete time-invariant system given by

$$(S)\begin{cases} x(t+1) &= \mathcal{A}x(t) \\ y(t) &= Cx(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $y(t) \in \mathbb{R}^p$ is the mesured output, $\mathcal{A} \in \mathcal{M}_n(\mathbb{R})$

and $C \in \mathcal{M}_{p,n}(\mathbb{R})$ are real constant matrices. It is assumed that the dynamics of the system, is not well known. The objective is the conception of Luenberger observer to estimate the state of the system. The matrix $\mathcal{A}$ can be represented as $\mathcal{A} = A + F$, with A matrix supposed to be known and F the unknown part of dynamic.

3 Main Results

3.1 Case 1: Dynamic with a single unknown element

We suppose that $\mathcal{A} = A + \alpha F$, where: $|\alpha| < 1$,

$$\mathrm{A} = \begin{pmatrix} 0 & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ \vdots & \vdots & \ddots & \vdots \\ \times & \times & \cdots & \times \end{pmatrix} \text{ and } \mathrm{F} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ so, the system (S)}$$

can be written

$$\begin{cases} x(t+1) &= Ax(t) + \alpha Fx(t) \\ y(t) &= Cx(t) \end{cases} \tag{2}$$

Our aim is to design a state observer described by

$$(Obs)\begin{cases} z(t+1) &= Az(t) + L(y(t) - \widehat{y}(t)) \\ \widehat{y}(t) &= Cz(t) \end{cases} \tag{3}$$

with $A - LC$ assumed to be stable.
Once introduced the error vector

$$e(t) = z(t) - Tx(t) \tag{4}$$

It is a matter of simple computions to show that the error vector updates according to the following equation

$$e(t+1) = \Big(A - LC\Big)e(t) + \Big[\Big(A - LC\Big)T + LC - TA - \alpha TF\Big]x(t) \tag{5}$$

We introduced a new variable

$$X(t) = \begin{bmatrix} e(t) \\ x(t) \end{bmatrix} \tag{6}$$

Then (5) becomes

$$\begin{aligned} X(t+1) = &\begin{bmatrix} A-LC & (A-LC)T+LC-TA \\ 0 & A-LC \end{bmatrix} X(t) \\ &+ \begin{bmatrix} 0 & -\alpha TF \\ 0 & LC+\alpha F \end{bmatrix} X(t) \end{aligned} \tag{7}$$

we can rewrite the uncertainty matrix $\begin{bmatrix} 0 & -\alpha TF \\ 0 & LC+\alpha F \end{bmatrix}$ as follows

$$\begin{bmatrix} 0 & -\alpha TF \\ 0 & LC+\alpha F \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -T \\ LC & I \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} I & 0 \\ 0 & \alpha F \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} 0 & I \\ 0 & I \end{bmatrix}}_{E} \tag{8}$$

which gives

$$X(t+1) = \overline{A}X(t) + D\Delta EX(t) \tag{9}$$

where

$$\overline{A} = \begin{bmatrix} A-LC & (A-LC)T+LC-TA \\ 0 & A-LC \end{bmatrix}$$

Since $|\alpha| < 1$ it is obvious that $\Delta'\Delta \leq I$, then we use the following result to establish the efficiency of our observer.

Theorem 3.1 [7], [21] *Let Q a positive definite symmetric matrix of appropriate dimension. The system (9) with $\Delta'\Delta \leq I$ is quadratically stable if and only if there exist $\epsilon > 0$ and a positive definite symmetric matrix $P \in \mathbb{R}^{n\times n}$ satisfying the following discrete Riccati equation:*

$$\overline{A}'(P^{-1} - \epsilon DD')^{-1}\overline{A} - P + \frac{1}{\epsilon}E'E + Q = 0$$

$$\textit{with } \frac{1}{\epsilon}I - D'PD > 0$$

It is possible to write a necessary and sufficient condition, in the form of LMI

Corollary 3.2 *The system (9) with $\Delta'\Delta \leq I$ is quadratically stable if and only if there exist $\epsilon > 0$ and a positive definite symmetric matrix $W \in \mathbb{R}^{n\times n}$ satisfying the following LMI:*

$$\begin{pmatrix} -W+\epsilon DD' & \overline{A}W & 0 \\ W\overline{A}' & -W & WE' \\ 0 & EW & -\epsilon I \end{pmatrix} < 0$$

Example 3.3 *Consider the uncertain system with parameters as follows*

$$\mathcal{A} = \begin{pmatrix} \alpha & 1.15 \\ -0.3 & -0.6 \end{pmatrix},\ L = \begin{bmatrix} -0.1 & 0.1 \end{bmatrix}',\ C = \begin{bmatrix} 1 & 2 \end{bmatrix} \text{ and } T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

We use the Matlab LMI Control Toolbox, and we obtain the solution as follows

$$W = 10^8 \times \begin{bmatrix} 3.0272 & -0.8923 & 0.0000 & 0.0000 \\ -0.8923 & 1.1608 & -0.0000 & -0.0000 \\ 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0.0000 & -0.0000 & -0.0000 & 0.0000 \end{bmatrix}$$

$$\epsilon = 9.0689 \times 10^{-17}$$

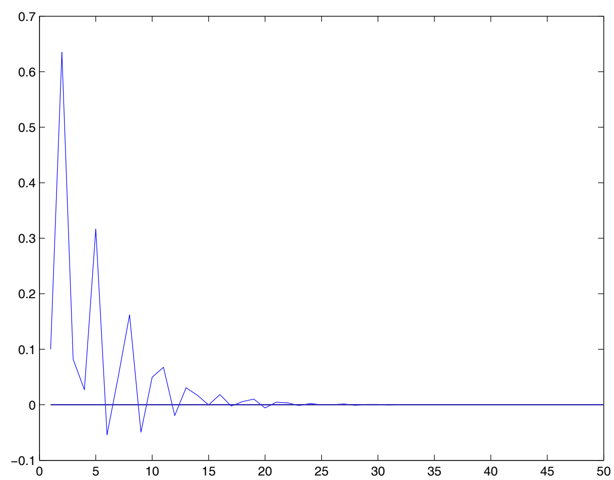

Figure 1: error For $\alpha = -0.5$

Example 3.4 *Consider the uncertain system with parameters as follows*

$$\mathcal{A} = \begin{pmatrix} \alpha & 0.2 \\ -1.8 & -0.25 \end{pmatrix},\ L = \begin{bmatrix} -0.01 & -0.001 \end{bmatrix}',\ C = \begin{bmatrix} -5 & -1 \end{bmatrix} \text{ and } T = I.$$

We use the Matlab LMI Control Toolbox, and we obtain

$$W = 10^8 \times \begin{bmatrix} 1.3191 & -0.3174 & 0.0000 & -0.0000 \\ -0.3174 & 6.2695 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & -0.0000 & 0.0000 \end{bmatrix}$$

$$\epsilon = 5.0411 \times 10^{-14}$$

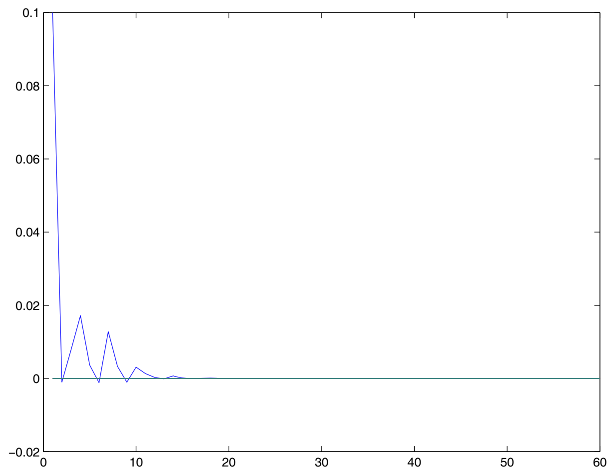

Figure 2: error For $\alpha = 0.1$

3.2 Case 2: Dynamic with an unknown colomn

In this case, we suppose that $\mathcal{A} = A + F$,

where A = $\begin{pmatrix} 0 & \times & \cdots & \times \\ 0 & \times & \cdots & \times \\ \vdots & \times & \ddots & \vdots \\ 0 & \times & \cdots & \times \end{pmatrix}$ and F = $\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}$

We follow the same approach as in the first case, we obtain

$$X(t+1) = \overline{A}X(t) + D\Delta EX(t) \tag{10}$$

The condition $\Delta'\Delta \leq I$ is satisfied if and only if $1 - \sum_{i=1}^{i=n} a_i^2 \geq 0$, consequently if $1-\sum_{i=1}^{i=n} a_i^2 \geq 0$, the system (10) is quadratically stable if and only if the LMI

$$\begin{pmatrix} -W+\epsilon DD' & \overline{A}W & 0 \\ W\overline{A}' & -W & WE' \\ 0 & EW & -\epsilon I \end{pmatrix} < 0$$

has symetric positive defined solution W and $\epsilon > 0$.

Example 3.5 *Consider the uncertain system with parameters as follows*

$\mathcal{A} = \begin{pmatrix} \alpha & 0.1 \\ \beta & -0.4 \end{pmatrix}$, $L = \begin{bmatrix} -0.1 & 0.1 \end{bmatrix}'$
$C = \begin{bmatrix} 2 & 4 \end{bmatrix}$, $T = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

We use the Matlab LMI Control Toolbox, and we obtain the solution

$$W = 10^8 \times \begin{bmatrix} 6.3671 & -1.5608 & -0.0000 & -0.0000 \\ -1.5608 & 5.6309 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 \end{bmatrix}$$

$$\epsilon = 2.8590 \times 10^{-11}$$

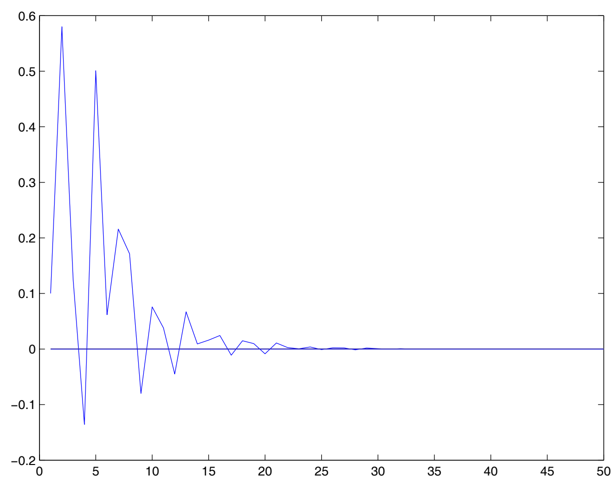

Figure 3: error For $\alpha = -0.5$ and $\beta = -0.3$

Example 3.6 *Consider the uncertain system with parameters as follows*

$$\mathcal{A} = \begin{pmatrix} \alpha & -0.3 \\ \beta & -0.2 \end{pmatrix},\ L = \begin{bmatrix} -0.01 & 0.1 \end{bmatrix}'$$
$$C = \begin{bmatrix} 0.2 & 1 \end{bmatrix},\ T = I$$

We use the Matlab LMI Control Toolbox, and we obtain the solution

$$W = 10^8 \times \begin{bmatrix} 4.2903 & 0.0781 & -0.0000 & -0.0000 \\ 0.0781 & 4.0008 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 \end{bmatrix}$$

$$\epsilon = 6.0985 \times 10^{-12}$$

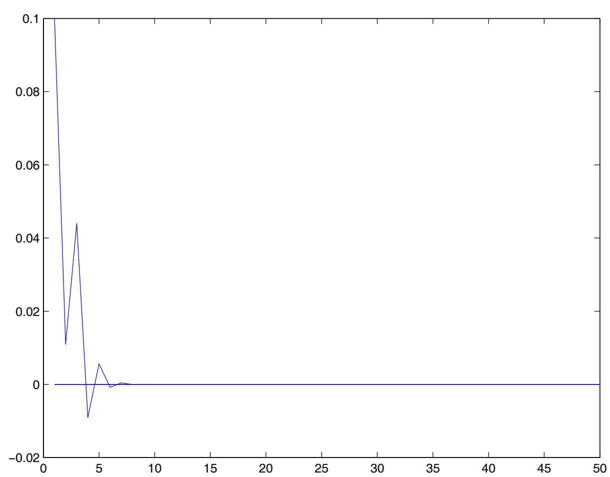

Figure 4: error For $\alpha = -0.2$ and $\beta = 0.22$

3.3 Case 3: Dynamic with unknown diagonal

We suppose that $\mathcal{A} = A + F$, where A $= \begin{pmatrix} 0 & \times & \cdots & \times \\ \times & 0 & \cdots & \times \\ \vdots & \times & \ddots & \vdots \\ \times & \times & \cdots & 0 \end{pmatrix}$

and F $= \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$

As, for previous cases we have

$$X(t+1) = \overline{A}X(t) + D\Delta EX(t) \tag{11}$$

The condition $\Delta'\Delta \leq I$ is satisfied if and only if $1 - a_i^2 \geqslant 0$ for each $i = 1, ..., n$, consequently if $1 - a_i^2 \geqslant 0$ for each $i = 1, ..., n$, the system (11) is quadratically stable if and only if the LMI

$$\begin{pmatrix} -W + \epsilon DD' & \overline{A}W & 0 \\ W\overline{A}' & -W & WE' \\ 0 & EW & -\epsilon I \end{pmatrix} < 0$$

has symetric positive defined solution W and $\epsilon > 0$.

Example 3.7 *Consider the uncertain system given by*

$$\mathcal{A} = \begin{pmatrix} \alpha & -0.5 \\ -1.8 & \beta \end{pmatrix},\ L = \begin{bmatrix} -2.1 & 0.12 \end{bmatrix}',\ C = \begin{bmatrix} -0.03 & 0.5 \end{bmatrix},\ T = \begin{pmatrix} 1 & -2 \\ 0 & -3 \end{pmatrix}$$

We use the Matlab LMI Control Toolbox, and we obtain the solution as follows

$$W = 10^8 \times \begin{bmatrix} 0.7179 & 0.0020 & -0.0000 & 0.0000 \\ 0.0020 & 2.3449 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0.0000 & -0.0000 & -0.0000 & 0.0000 \end{bmatrix}$$

$$\epsilon = 8.7013 \times 10^{-13}$$

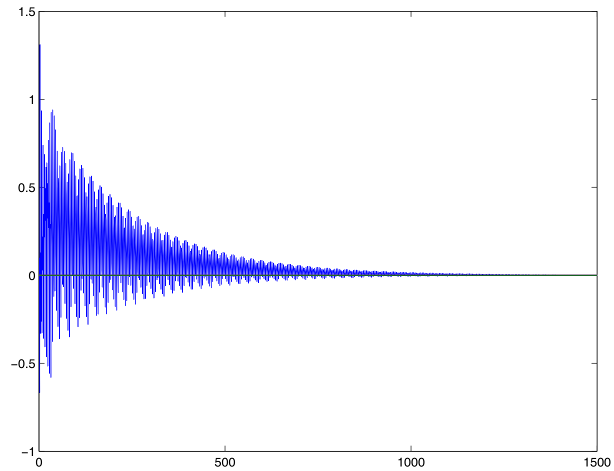

Figure 5: error For $\alpha = -0.01$ and $\beta = 0.02$

Example 3.8 *Consider the uncertain system given by*

$$\mathcal{A} = \begin{pmatrix} \alpha & -5 \\ -0.08 & \beta \end{pmatrix},\ L = \begin{bmatrix} -0.21 & 0.12 \end{bmatrix}',\ C = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix}\ ,\ T = I$$

We use the Matlab LMI Control Toolbox, and we obtain the solution as follows

$$W = 10^8 \times \begin{bmatrix} 2.3312 & 0.0278 & 0.0000 & -0.0000 \\ 0.0278 & 0.0736 & -0.0000 & 0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$\epsilon = 1.9108 \times 10^{-13}$$

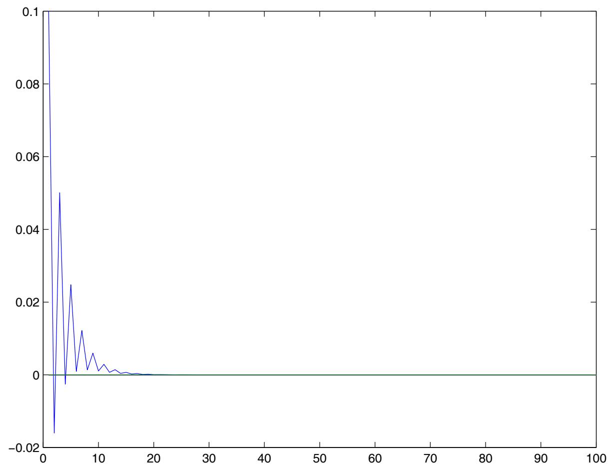

Figure 6: error For $\alpha = -0.1$ and $\beta = 0.05$

3.4 Case 4: Dynamic with several unknown elements

We suppose that $\mathcal{A} = A + F$, where $A = \begin{pmatrix} 0 & \times & \cdots & \times \\ \times & \times & \cdots & 0 \\ \vdots & \times & \ddots & \vdots \\ \times & 0 & \cdots & \times \end{pmatrix}$

and $F = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & a_r \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_2 & \cdots & 0 \end{pmatrix}$

in this case the dynamics can be represented as $\mathcal{A} = A + \sum_{i=1}^{i=r} F_i$ where F_i is the matrix where an element of the i^{th} column is unknown.
We follow the same approach as in the first case, we obtain

$where\overline{A}, D_i$ and E_i are known matrices.
Under the condition $\Delta_i'\Delta_i \leq I$ for each $i = 1, ..., r$ we have the following result

Theorem 3.9 [21] *Let Q a positive definite symmetric matrix of appropriate dimension. The system (12) is quadratically stable if there exist a set of scalar $\epsilon_i > 0, i = 1, ..., r$ and a positive definite symmetric matrix P satisfying the following discrete Riccati equation:*

$$\overline{A}'(P^{-1} - DD')^{-1}\overline{A} - P + E'E + Q = 0$$

$$with\ \ I - D'PD > 0$$

where D *is the bloc matrix given by* $\begin{bmatrix}\sqrt{\epsilon_1}D_1 & ... & \sqrt{\epsilon_r}D_r\end{bmatrix}$ *and* $E' = \begin{bmatrix}\frac{1}{\sqrt{\epsilon_1}}E'_1 & ... & \frac{1}{\sqrt{\epsilon_r}}E'_r\end{bmatrix}$.

It is possible to write a sufficient condition in the form of LMI

Corollary 3.10 *The system* (12) *with* $\Delta'_i\Delta_i \leq I$ *for each* $i = 1, ..., r$ *is quadratically stable if there exist* $\epsilon > 0$ *and a positive definite symmetric matrix* $W \in \mathbb{R}^{n\times n}$ *satisfying the following LMI:*

$$\begin{pmatrix} -W + \widetilde{D}\epsilon\widetilde{D}' & \overline{A}W & 0 \\ W\overline{A}' & -W & W\widetilde{E}' \\ 0 & \widetilde{E}W & -\epsilon \end{pmatrix} < 0$$

where $\epsilon = \begin{pmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_r \end{pmatrix}$, $\widetilde{D} = \begin{bmatrix} D_1 & ... & D_r \end{bmatrix}$ *and* $\widetilde{E} = \begin{bmatrix} E_1 \\ \vdots \\ E_r \end{bmatrix}$

Example 3.11 *Consider the uncertain system given by*

$$\mathcal{A} = \begin{pmatrix} -0.1 & -0.007 & \alpha & 0.3 \\ \beta & -1 & 0.2 & -1 \\ -1 & \gamma & -1 & -0.25 \\ 0.8 & 0.1 & 0.5 & \delta \end{pmatrix},\ L = \begin{bmatrix} -0.1 & -0.1 & 0.1 & -0.1 \end{bmatrix}',$$

$$C = \begin{bmatrix} -0.5 & 0.1 & 0.2 & -0.004 \end{bmatrix} \text{ and } T = \begin{pmatrix} 2 & 0 & 0.5 & 0.1 \\ 0 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

We use the Matlab LMI Control Toolbox, and we obtain the solution as follows

$$W = 10^{-8}\begin{bmatrix} 0.9639 & -0.1845 & -1.0832 & -0.0272 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ -0.1845 & 5.1056 & -0.3356 & -1.1323 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ -1.0832 & -0.3356 & 1.3496 & 0.1306 & -0.0000 & -0.0000 & 0.0000 & -0.0000 \\ -0.0272 & -1.1323 & 0.1306 & 1.3040 & 0.0000 & 0.0000 & 0.0000 & -0.0000 \\ 0.0000 & -0.0001 & -0.0001 & 0.0008 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ -0.0000 & 0.0001 & -0.0001 & -0.0002 & 0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0001 & -0.0000 & -0.0009 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0001 & -0.0006 & 0.0002 & 0.0002 & 0.0000 & -0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$\epsilon = 10^{-13} \times \begin{pmatrix} 0.1050 & 0 & 0 & 0 \\ 0 & 0.6806 & 0 & 0 \\ 0 & 0 & 0.0006 & 0 \\ 0 & 0 & 0 & 0.0006 \end{pmatrix}$$

4 Conclusions and prospects

In this paper, the design problem of observers for a class of finite dimensional linear discrete time-invariant systems in which the dynamics are partially unknown is studied. Sufficient conditions to its existence are presented. Future research will focus on the delayed systems case.

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