International Mathematical Forum, 5, 2010, no. 44, 2173 - 2189

The Pseudo-Differential Operator $h_{\mu,a}$ on Some Ultradifferentiable Spaces

Akhilesh Prasad and Manish Kumar

Department of Applied Mathematics
Indian School of Mines, Dhanbad-826004, India
apr_bhu@yahoo.com, manish.math.bhu@gmail.com

Abstract

The pseudo-differential operator (p.d.o.) $h_{\mu,a}$ associated with Bessel operator involving the symbol $a(x, y)$ whose derivatives satisfy growth conditions depending on some increasing sequences is studied on certain ultradifferentiable function spaces. It is shown that the oprator $h_{\mu,a}$ is continuous linear map of ultradifferentiable spaces into another ultradifferentiable space. A special (p.d.o.) called the Hankel potential is defined and some of its properties are investigated.

Mathematics Subject Classification: 46F12; 47A05; 47G05

Keywords: Pseudo-differential operator; Hankel transformation; Hankel translation; Hankel convolution; Hankel potential

1 Introduction

The pseudo-differential operator (p.d.o.) associated with the Bessel operator have applications in the study of boundary value problems on the half line. The p.d.o. $h_{\mu,a}$, was introduced by Pathak and Prasad [4], and its properties were investigated using Zemanian's theory of the Hankel transformation. Zemanian's theory was further extended by Lee [3], to certain spaces of ultradistributions. For this purpose, the spaces $H_{\mu,a_k,A}$, $H_{\mu}^{b_q,B}$ and $H_{\mu,a_k,A}^{b_q,B}$ of ultradifferentiable functions were defined as follows. Similar spaces have been studied in [2] and [6]. Wong [7] has studied pseudo-differential operators and Bessel potential involving Fourier transform and Rodino [6] has discussed p.d.o. on Gevrey spaces (ultradifferentiable function spaces), which are intermediate spaces between the spaces of $\mathbb{C}^{\infty}$-functions and analytic functions. Recently in this paper we have studied the properties of p.d.o. and Hankel potential involving Hankel transformation.

Zemanian's [8], introduced the function space H_μ consisting of all complex valued infinitely differentiable function ϕ defined on $I=(0,\infty)$ satisfying

$$\gamma^{\mu}_{m,k}(\phi)=\sup_{x\in I}\left|x^m(x^{-1}d/dx)^k x^{-\mu-1/2}\phi(x)\right|<\infty, \qquad \forall\, m,k\in\mathbb{N}_0. \tag{1}$$

The pseudo-differential operator involving the symbol $a(x,y)$ is defined by

$$(h_{\mu,a}\phi)(x)=\int_0^\infty (xy)^{1/2}J_\mu(xy)a(x,y)\widehat{\phi}(y)dy, \qquad \mu\geq -1/2, \tag{2}$$

where $\widehat{\phi}$ is the Hankel transformation defined by

$$\widehat{\phi}(y)=(h_\mu\phi)(y)=\int_0^\infty (xy)^{1/2}J_\mu(xy)\phi(x)dx,\ \phi\in H_\mu(I) \tag{3}$$

and the pseudo-differential operator involving the symbol $a(y)=(1+y^2)^{-s/2}, s\in\mathbb{R}$ is defined by

$$\begin{aligned}(h_{\mu,a}\phi)(x) &= \int_0^\infty (xy)^{1/2}J_\mu(xy)a(y)\widehat{\phi}(y)dy, \qquad \mu\geq -1/2\\ &= h_\mu^{-1}\left[(1+y^2)^{-s/2}\widehat{\phi}\right](x) \end{aligned} \tag{4}$$

and will be called Hankel potential of order s, it is denoted by h^s_μ, where $\widehat{\phi}$ is as (3) and J_μ is the Bessel function of the first kind of order μ. We shall study the properties of the symbol $a(x,y)$ and $a(y)=(1+y^2)^{-s/2}$, $s\in\mathbb{R}$ in section 2 and section 3 respectively. The following definitions and results will be needed in the sequel.

The space $L^p_\mu(\mu\geq -1/2)$ is the set of all measurable functions ϕ on $I=(0,\infty)$ such that

$$\|\phi\|^p_\mu=\int_0^\infty |\phi(x)|^p\, x^{\mu+1/2}dx<\infty. \tag{5}$$

The Hankel translation of $\phi\in L^1_\mu(I)$ is defined by

$$(\tau_z\phi)(w)=\int_0^\infty \phi(y)D_\mu(y,w,z)dy, \qquad \forall\, w,z\in I, \tag{6}$$

where

$$D_\mu(y,w,z)=\int_0^\infty t^{-\mu-1/2}j_\mu(yt)j_\mu(wt)j_\mu(zt)dt \tag{7}$$

and

$$j_\mu(wt) = (wt)^{1/2} J_\mu(wt). \tag{8}$$

The Hankel convolution transform of two functions $\phi, \psi \in L^1_\mu(I)$ is defined by

$$(\phi \# \psi)(z) = \int_0^\infty \phi(w)(\tau_z \psi)(w) dw, \qquad a.e. \quad z \in I, \tag{9}$$

we shall also make use of the following results [[1],p. 285]

$$h_\mu(\tau_z \phi)(u) = u^{-\mu-1/2} j_\mu(uz)(h_\mu \phi)(u), \qquad \forall\ u, z \in I \tag{10}$$

and

$$h_\mu(\phi \# \psi)(u) = u^{-\mu-1/2}(h_\mu \phi)(u)(h_\mu \psi)(u), \qquad \forall\ u \in I. \tag{11}$$

We shall use the notation and terminology of [5, 8]. The differential operators N_μ, M_μ and S_μ are defined by

$$N_\mu = N_{\mu,x} = x^{\mu+1/2}(d/dx)x^{-\mu-1/2}, \tag{12}$$

$$M_\mu = M_{\mu,x} = x^{-\mu-1/2}(d/dx)x^{\mu+1/2}, \tag{13}$$

$$S_\mu = S_{\mu,x} = M_\mu N_\mu = d^2/dx^2 + (1 - 4\mu^2)/4x^2. \tag{14}$$

We have the following relations for any $\phi \in H_\mu$:

$$h_{\mu+1}(-x\phi) = N_\mu h_\mu \phi, \tag{15}$$

$$h_{\mu+1}(N_\mu \phi) = -y h_\mu \phi, \tag{16}$$

$$h_\mu(S_\mu \phi) = -y^2 h_\mu \phi \tag{17}$$

and

$$S^r_\mu \phi(x) = \sum_{j=0}^{r} b_j x^{2j+\mu+1/2}(d/dx)^{r+j}(x^{-\mu-1/2}\phi(x)), \tag{18}$$

where the b_j are constants depending on μ.

The following formula are given in [[8],pp.129,134].

$$(x^{-1}d/dx)^k(x^{-\mu-1/2}\psi\phi) = \sum_{\nu=0}^{k} \binom{k}{\nu} (x^{-1}d/dx)^\nu \psi (x^{-1}d/dx)^{k-\nu}(x^{-\mu-1/2}\phi) \tag{19}$$

$$(x^{-1}d/dx)^k(x^{-\mu}J_\mu(x)) = (-1)^k(x)^{-(\mu+k)}J_{\mu+k}(x) \tag{20}$$

$$(x^{-1}d/dx)^k(x^{\mu}J_\mu(x)) = (x)^{\mu-k}J_{\mu-k}(x). \tag{21}$$

Let $\{a_k\}_{k\in\mathbb{N}_0}$ and $\{b_q\}_{q\in\mathbb{N}_0}$ be arbitrary sequences of positive numbers which satisfy the following conditions

$$a_k^2 \le a_{k-1}a_{k+1}, \qquad \forall\ k \ge 1, \tag{22}$$

$$b_q^2 \le b_{q-1}b_{q+1}, \qquad \forall\ q \ge 1. \tag{23}$$

Immediate consequences of these inequalities are

$$a_p a_k \le a_0 a_{p+k}, \qquad \forall\ p,k = 0,1,2,\ldots \tag{24}$$

$$b_p b_q \le b_0 b_{p+q}, \qquad \forall\ p,q = 0,1,2,\ldots, \tag{25}$$

from inequality (22) it can be proved that

$$(a_k/a_{k+1}) \le (a_{k-1}/a_k) \le (a_{k-2}/a_{k-1}) \cdots \le (a_0/a_1) \tag{26}$$

and

$$\begin{aligned} a_{k-r} &= (a_{k-r}/a_{k-r+1}) \times (a_{k-r+1}/a_{k-r+2}) \times \cdots (a_{k+1}/a_k) \times a_k \\ &\le (a_0/a_1) \times (a_0/a_1) \times \cdots (a_0/a_1) \times a_k; \end{aligned}$$

so that

$$a_{k-r} \le (a_0/a_1)^r \times a_k. \tag{27}$$

Furthermore, assume that there are constants $R_1, R_2 > 0$ and $H_1, H_2 > 1$ such that

$$a_p \le R_1 H_1^p \min_{0\le q\le p} a_q a_{p-q}, \qquad \forall\ p,q \in \mathbb{N}_0, \tag{28}$$

$$b_p \le R_2 H_2^p \min_{0\le q\le p} b_q b_{p-q}, \qquad \forall\ p,q \in \mathbb{N}_0. \tag{29}$$

Let the constants c_1, h_1, c_2, h_2, c and h be such that for all $k, q \in \mathbb{N}_0$,

$$a_{k+1} \le c_1 h_1^k a_k, \tag{30}$$

$$b_{q+1} \le c_2 h_2^q b_q, \tag{31}$$

$$b_{q+1} \geq ch^q b_q. \tag{32}$$

The conditions (30) and (31) may be replaced by the following stronger conditions whenever necessary

$$a_{r+k} \leq L_1 R_1^{k+r} a_r a_k, \qquad \forall\ r, k \geq 0, \tag{33}$$

$$b_{r+q} \leq L_2 R_2^{r+q} b_r b_q, \qquad \forall\ r, q \geq 0, \tag{34}$$

where L_1, R_1, L_2 and R_2 are positive constants.
The spaces of type H_μ, that is $H_{\mu,a_k,A}$, $H_\mu^{b_q,B}$ and $H_{\mu,a_k,A}^{b_q,B}$ are defined as follows:

Definition 1.1 *Let ϕ be infinitely differentiable function on I. Then $\phi \in H_{\mu,a_k,A}$ if and only if*

$$\|\phi\|_q^\mu = \sup_{k\in\mathbb{N}_0} \sup_{x\in I} \frac{\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\phi(x)\right|}{(A+\sigma)^k a_k} < \infty$$

for every $q \in \mathbb{N}_0$ where A is a certain positive constant depending on ϕ and $\sigma > 0$ is arbitrary.

Definition 1.2 *The space $H_\mu^{b_q,B}$ is defined as follows: $\phi \in H_\mu^{b_q,B}$ if and only if*

$$\|\phi\|_k^\mu = \sup_{q\in\mathbb{N}_0} \sup_{x\in I} \frac{\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\phi(x)\right|}{(B+\rho)^q b_q} < \infty$$

for every $k \in \mathbb{N}_0$ where B is a positive constant depending on ϕ and $\rho > 0$ is arbitrary.

Definition 1.3 *The function $\phi \in H_{\mu,a_k,A}^{b_q,B}$ if and only if*

$$\|\phi\|^\mu = \sup_{k,q\in\mathbb{N}_0} \sup_{x\in I} \frac{\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\phi(x)\right|}{(A+\sigma)^k a_k (B+\rho)^q b_q} < \infty$$

where σ and ρ are as above and A and B are certain positive constants depending on ϕ.

The elements of the spaces $H_{\mu,a_k,A}$, $H_\mu^{b_q,B}$ and $H_{\mu,a_k,A}^{b_q,B}$ are called ultradifferentiable functions and those of the corresponding dual spaces $(H_{\mu,a_k,A})'$, $(H_\mu^{b_q,B})'$ and $(H_{\mu,a_k,A}^{b_q,B})'$ are called ultradifferentiable.

From [5], we have the following Theorem:

Theorem 1.4 *If $\{a_k\}$ satisfies (22) and (33) and $\{b_q\}$ satisfies (34) $\forall\ k, q \in \mathbb{N}_0$, then for each fixed z, $0 < z < z_0, \mu \geq -1/2$, the mapping $\phi \longmapsto \tau_z\phi$ is continuous* *from the spaces*

(i) $H^{b_q,B}_{\mu,a_k,A}$ *into* $H^{a_q^2 b_q^2,B_4}_{\mu,a_k^3 b_k,A_3}$, *where* $A_3 = A_1 B_3 (R^{\otimes})^2, B_3 = R_1^2[B_1 + (z_0 a_0/a_1)^2]$, $B_1 = A^2(R^*)^6, A_1 = AB(R^*)^2, B_4 = A_1^2(R^{\otimes})^6, R^* = \max(1, R_1), R^{\otimes} = \max(1, R_1 R_2)$ *and* R_1, R_2 *are defined by (33) and (34), and*

(ii) $H_{\mu,a_k,A}$ *into* H_{μ,a_k^2,A_2}, *where* $A_2 = R_1^2[B_1 + (z_0 a_0/a_1)^2]$, B_1 *and* R^* *as above.*

Hankel transformation cannot be defined on the whole of the space $H^{b_q,B}_{\mu}$; but it could be defined on a certain subspace $\widetilde{H}^{b_q,B}_{\mu}$ of $H^{b_q,B}_{\mu}$ in which the following condition is satisfied

$$\sup_k Q^{\mu}_{k+2q} = Q^{*\mu}_q, \tag{35}$$

where $Q^{*\mu}_q$ are constants restraining the ϕ's in $H^{b_q,B}_{\mu}$. Then

(iii) $\widetilde{H}^{b_q,B}_{\mu}$ into $H^{b_q^2,B_2}_{\mu}$, where $B_2 = B^2(R^*)^6$ and R^* as above.

2 Pseudo-differential operator involving Hankel translation and Hankel convolution of the spaces of type H_μ

Definition 2.1 *The symbol* $a(x,y)$ *is defined to be a complex valued function belonging to the space* $\mathbf{C}^\infty(I \times I)$, *such that its derivatives satisfy the growth condition*

$$\left|(x^{-1}d/dx)^{\alpha}(y^{-1}d/dy)^{\beta}a(x,y)\right| \leq L_m(C+\delta)^{\alpha}c_{\alpha}(D+\eta)^{\beta}d_{\beta}(1+y)^{m-\beta} \tag{36}$$

for all $\alpha, \beta \in \mathbb{N}_0, \delta > 0, \eta > 0$ *and* $L_m > 0$, *where* m *is a fixed real number and* $\{c_\alpha\}$ *and* $\{d_\beta\}$ *are certain sequences of positive real numbers satisfying some of the conditions of type (22)-(32). The set of all such symbols will be denoted* $S^m_{c_\alpha,d_\beta}$.

We have the following two interesting theorems [5].

Theorem 2.2 *If* $\{a_k\}$ *and* $\{b_q\} \forall\ k, q \in \mathbb{N}_0$ *satisfies (22) and (23) respectively then for* $\mu \geq -1/2$, *the mapping* $(\phi, \psi) \longmapsto (\phi \# \psi)$ *is linear and continuous from the spaces*

(i) $H_{\mu,a_k,A} \times H_{\mu,a_k,A}$ *into* $\widetilde{H}_{\mu,a_k^2,B_1}$, *where* $B_1 = A^2(R^*)^6$ *and* $R^* = \max(1, R_1)$

(ii) $\widetilde{H}^{b_q,B}_{\mu} \times \widetilde{H}^{b_q,B}_{\mu}$ *into* $H^{b_q^2,B_2}_{\mu}$, *where* $B_2 = B^2(R^*)^6$ *and* R^* *as above*

(iii) $H^{b_q,B}_{\mu,a_k,A} \times H^{b_q,B}_{\mu,a_k,A}$ *into* $H^{a_q^2 b_q^2,B_5}_{\mu,a_k^3 b_k,A_4}$*, where* $A_4 = (R^{\otimes})^2 A_1 B_1, B_5 = (R^{\otimes})^6 A_1^2$*,* $R^{\otimes} = \max(1, R_1 R_2), A_1 = AB(R^*)^2$ *and* B_1 *as above.*

Theorem 2.3 *If* $\{a_k\}$ *and* $\{b_q\}\, \forall\, k, q \in \mathbb{N}_0$ *satisfies (22) and (23) respectively then for* $\mu \geq -1/2$*, the mapping* $(\phi, \psi) \longmapsto h_\mu(\phi \# \psi)$ *is continuous linear mapping from*

(i) $H_{\mu,a_k,A} \times H_{\mu,a_k,A}$ *into* $H^{a_q^2,B_1}_{\mu}$*, where* $B_1 = A^2(R^*)^6$ *and* $R^* = \max\{1, R_1\}$

(ii)$\widetilde{H}^{b_q,B}_{\mu} \times \widetilde{H}^{b_q,B}_{\mu}$ *into* $H_{\mu,b_k,B}$ *and*

(iii) $H^{b_q,B}_{\mu,a_k,A} \times H^{b_q,B}_{\mu,a_k,A}$ *into* $H^{a_q^2,B_1}_{\mu,a_k b_k,A_1}$*, where* $A_1 = AB(R^*)^2, B_1 = A^2(R^*)^6$ *and* $R^* = \max\{1, R_1\}$*.*

Theorem 2.4 *If* $\{a_k\}$ *and* $\{b_q\}\, \forall\, k, q \in \mathbb{N}_0$ *satisfies (22) and (23) respectively then for each fixed* $z, 0 < z < z_0$ *and* $\mu \geq -1/2$*, the mapping* $(\phi, \psi) \longmapsto h_\mu \tau_z(\phi \# \psi)$ *is linear and continuous from*

(i) $H_{\mu,a_k,A} \times H_{\mu,a_k,A}$ *into* $H^{a_q^2,B_3}_{\mu}$*, where* $B_3 = R^2[B_1 + (z_0 a_0/a_1)^2], B_1 = A^2(R^*)^6$ *and* $R^* = \max(1, R_1)$

(ii) $\widetilde{H}^{b_q,B}_{\mu} \times \widetilde{H}^{b_q,B}_{\mu}$ *into* $H_{\mu,b_k,B}$ *and*

(iii) $H^{b_q,B}_{\mu,a_k,A} \times H^{b_q,B}_{\mu,a_k,A}$ *into* $H^{a_q^2,B_3}_{\mu,a_k b_k,A_1}$*, where* $A_1 = AB(R^*)^2$ *and* B_3, B_1 *are as above and* R_1 *are determined by (33).*

Proof: Here we prove (i). The other two parts can be proved in similar way. Let $(\phi, \psi) \in H_{\mu,a_k,A} \times H_{\mu,a_k,A}$. Then applying Leibnitz type formula (19) and using (10), (33) and (22) we obtain

$$\begin{aligned}
&\gamma^{\mu}_{k,q}[h_\mu \tau_z(\phi \# \psi)(x)] = \sup_{x \in I} \left| x^k (x^{-1} d/dx)^q x^{-\mu-1/2} (h_\mu \tau_z(\phi \# \psi))(x) \right| \\
&= \sup_{x \in I} \left| x^k (x^{-1} d/dx)^q x^{-2\mu-1} j_\mu(xz) (h_\mu(\phi \# \psi))(x) \right| \\
&= \sup_{x \in I} \left| x^k (x^{-1} d/dx)^q x^{-2\mu-1} (xz)^{1/2} J_\mu(xz) (h_\mu(\phi \# \psi))(x) \right| \\
&= \sum_{r=0}^{q} \binom{q}{r} \sup_{x \in I} \left| x^k (x^{-1} d/dx)^{q-r} x^{-\mu-1/2} (h_\mu(\phi \# \psi))(x) \right| \\
&\times \sup_{x \in I} \left| z^{1/2} (x^{-1} d/dx)^r x^{-\mu} J_\mu(xz) \right| .
\end{aligned}$$

Using the theorem (2.3)(i), then we have

$$\gamma^{\mu}_{k,q}[h_\mu\tau_z(\phi\#\psi)(x)] = \sum_{r=0}^{q}\binom{q}{r}(B_1+\rho)^{q-r}a^2_{q-r}z^{\mu+2r+1/2}$$
$$\times\left\|(h_\mu(\phi\#\psi))(x)\right\|^{\mu}_{q}\sup_{z\in I}\left|(xz)^{-(\mu+r)}J_\mu(xz)\right|$$
$$\leq \sum_{r=0}^{q}\binom{q}{r}(B_1+\rho)^{q-r}a^2_{q-r}z^{\mu+2r+1/2}A_{\mu,r}\left\|(h_\mu(\phi\#\psi))\right\|^{\mu}_{q}$$
$$\leq A_\mu C z^{\mu+1/2}\sum_{r=0}^{q}\binom{q}{r}z^{2r}(B_1+\rho)^{q-r}a_0a_{2q-2r}\left\|(h_\mu(\phi\#\psi))\right\|^{\mu}_{q}$$
$$\leq A_\mu C a_0 z^{\mu+1/2}\sum_{r=0}^{q}\binom{q}{r}z^{2r}(B_1+\rho)^{q-r}(a_0/a_1)^{2r}a_{2q}\left\|(h_\mu(\phi\#\psi))\right\|^{\mu}_{q}$$
$$\leq A_\mu C a_0 z^{\mu+1/2}\sum_{r=0}^{q}\binom{q}{r}z^{2r}(B_1+\rho)^{q-r}(a_0/a_1)^{2r}L_1R_1^{2q}a_q^2\left\|(h_\mu(\phi\#\psi))\right\|^{\mu}_{q}$$
$$\leq A_\mu C a_0 L z_0^{\mu+1/2}R_1^{2q}[B_1+(z_0a_0/a_1)^2+\rho]^q a_q^2\left\|(h_\mu(\phi\#\psi))\right\|^{\mu}_{q}$$
$$\leq \acute{C}(B_3+\acute{\rho})^q a_q^2$$

where $B_3 = R_1^2[B_1+(z_0a_0/a_1)^2+\rho], A_\mu = max_{0\leq r\leq q}A_{\mu,r}, \acute{C} = A_\mu C a_0 L z_0^{\mu+1/2}$ and $B_1 = A^2(R^*)^6$;
so that $h_\mu\tau_z(\phi\#\psi)(x)\in H_\mu^{a_q^2,B_3}$.

Theorem 2.5 *If $\{a_k\},\{c_k\}$ and $\{d_k\}, k\in\mathbb{N}_0$, satisfying condition (27), for each fixed $z, 0<z<z_0$ and $\mu\geq -1/2$ and let the symbol $a(x,y)$ satisfy (36) then p.d.o. $(\phi,\psi)\longmapsto h_{\mu,a}\tau_z(\phi\#\psi)$ is continuous linear mapping from $H_{\mu,a_k,A}\times H_{\mu,a_k,A}$ into $\widetilde{H}_{\mu,a_k^{*2},A_3}$, where $A_3 = [(a_0^*/a_1^*)D+B_3], B_3 = R_1^2[B_1+(z_0a_0/a_1)^2+\rho]$ and $a_k^* = \max_{k\in N_0}(a_k,d_k)$.*

Proof: Let $(\phi,\psi)\in H_{\mu,a_k,A}\times H_{\mu,a_k,A}$, then by Theorem (2.3)(i) $h_\mu(\phi\#\psi)\in H_\mu^{a_q^2,B_1}$ and by Theorem (2.4)(i) $h_\mu\tau_z(\phi\#\psi)\in H_\mu^{a_q^2,B_3}$.

Now assume that

$$\begin{aligned}\Phi(x) &= (h_{\mu,a}\tau_z(\phi\#\psi))(x)\\ &= \int_0^\infty (xy)^{1/2}J_\mu(xy)a(x,y)(h_\mu\tau_z(\phi\#\psi))(y)dy.\end{aligned}$$

Using Zemanian's technique [[8],p. 144] we have

$$\begin{aligned} N_\mu\Phi(x) &= x^{\mu+1/2}(d/dx)x^{-\mu-1/2}\Phi(x) \\ &= x^{\mu+1+1/2}(x^{-1}d/dx)x^{-\mu-1/2}\Phi(x). \qquad (37) \\ N_{\mu+1}N_\mu\Phi(x) &= x^{\mu+1+1/2}(d/dx)x^{-(\mu+1)-1/2}N_\mu\Phi(x) \\ &= x^{\mu+2+1/2}(x^{-1}d/dx)x^{-\mu-3/2}[x^{\mu+3/2}(x^{-1}d/dx)x^{-\mu-1/2}\Phi(x)] \\ &= x^{\mu+2+1/2}(x^{-1}d/dx)^2x^{-\mu-1/2}\Phi(x). \end{aligned}$$

Similarly, using (19), we have

$$\begin{aligned} & N_{\mu+q-1}\ldots N_\mu\Phi(x) = x^{\mu+q+1/2}(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x) \qquad (38) \\ & = x^{\mu+q+1/2}(x^{-1}d/dx)^q x^{-\mu-1/2}\int_0^\infty (xy)^{1/2}J_\mu(xy)a(x,y)(h_\mu\tau_z(\phi\#\psi))(y)dy \\ & = x^{\mu+q+1/2}(x^{-1}d/dx)^q\int_0^\infty y^{1/2}x^{-\mu}J_\mu(xy)a(x,y)(h_\mu\tau_z(\phi\#\psi))(y)dy \\ & = x^{\mu+q+1/2}\int_0^\infty y^{1/2}\sum_{r=0}^{q}\binom{q}{r}(x^{-1}d/dx)^{q-r}x^{-\mu}J_\mu(xy) \\ & \times(x^{-1}d/dx)^r a(x,y)(h_\mu\tau_z(\phi\#\psi))(y)dy \\ & = x^{\mu+q+1/2}\int_0^\infty y^{1/2}\sum_{r=0}^{q}\binom{q}{r}(-y)^{q-r}x^{-\mu-q+r}J_{\mu+q-r}(xy) \\ & \times(x^{-1}d/dx)^r a(x,y)(h_\mu\tau_z(\phi\#\psi))(y)dy. \end{aligned}$$

Therefore,

$$\begin{aligned} & N_{\mu+q-1}\ldots N_\mu\Phi(x) = \sum_{r=0}^{q}\binom{q}{r}\int_0^\infty x^{r+1/2}y^{1/2}(x^{-1}d/dx)^r a(x,y) \\ & \times(h_\mu\tau_z(\phi\#\psi))(y)(-y)^{q-r}J_{\mu+q-r}(xy)dy \qquad (39) \\ & = \sum_{r=0}^{q}\binom{q}{r}x^r\int_0^\infty (xy)^{1/2}J_{\mu+q-r}(xy)[(x^{-1}d/dx)^r a(x,y)(-y)^{q-r}(h_\mu\tau_z(\phi\#\psi))(y)]dy \\ & = \sum_{r=0}^{q}\binom{q}{r}x^r h_{\mu+q-r}[(x^{-1}d/dx)^r a(x,y)(-y)^{q-r}(h_\mu\tau_z(\phi\#\psi))(y)](x). \qquad (40) \end{aligned}$$

Using formula $-x(h_\mu\phi) = h_{\mu+1}(N_\mu\phi)$ in (40), we get

$$
\begin{aligned}
&(-x)N_{\mu+q-1}\ldots N_\mu\Phi(x)\\
&=\sum_{r=0}^{q}\binom{q}{r}x^r(-x)h_{\mu+q-r}[(x^{-1}d/dx)^r a(x,y)(-y)^{q-r}(h_\mu\tau_z(\phi\#\psi))(y)](x)\\
&=\sum_{r=0}^{q}\binom{q}{r}x^r h_{\mu+q-r+1}N_{\mu+q-r}[(x^{-1}d/dx)^r a(x,y)(-y)^{q-r}(h_\mu\tau_z(\phi\#\psi))(y)](x)\\
&=\sum_{r=0}^{q}\binom{q}{r}x^r\int_0^\infty (xy)^{1/2}J_{\mu+q-r+1}(xy)\\
&\times N_{\mu+q-r}[(x^{-1}d/dx)^r a(x,y)(-y)^{q-r}(h_\mu\tau_z(\phi\#\psi))(y)]dy\\
&=\sum_{r=0}^{q}\binom{q}{r}\int_0^\infty x^r(xy)^{1/2}J_{\mu+q-r+1}(xy)y^{\mu+q-r+1/2}\\
&\times(d/dy)y^{-\mu-q+r-1/2}[(x^{-1}d/dx)^r a(x,y)(-y)^{q-r}(h_\mu\tau_z(\phi\#\psi))(y)]dy\\
&=\sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}\int_0^\infty x^{r+1/2}y^{\mu+q-r+2}(y^{-1}d/dy)[y^{-\mu-1/2}\\
&\times(h_\mu\tau_z(\phi\#\psi))(y)(x^{-1}d/dx)^r a(x,y)J_{\mu+q-r+1}(xy)]dy\\
&=\sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}x^r\int_0^\infty (xy)^{1/2}J_{\mu+q-r+1}(xy)\\
&\times[y^{\mu+q-r+1+1/2}(y^{-1}d/dy)\big\{y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)(x^{-1}d/dx)^r a(x,y)\big\}]dy\\
&=\sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}x^r h_{\mu+q-r+1}[y^{\mu+q-r+1+1/2}\\
&\times(y^{-1}d/dy)\big\{y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)(x^{-1}d/dx)^r a(x,y)\big\}]. \qquad (41)
\end{aligned}
$$

Now, from (41) again using result $-x(h_\mu\phi) = h_{\mu+1}(N_\mu\phi)$, we get

$$
\begin{aligned}
&(-x)^2(N_{\mu+q-1}\ldots N_\mu\Phi(x)) = \sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}x^r(-x)h_{\mu+q-r+1}\\
&\times[y^{\mu+q-r+1+1/2}(y^{-1}d/dy)\big\{y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)(x^{-1}d/dx)^r a(x,y)\big\}]\\
&=\sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}x^r h_{\mu+q-r+2}N_{\mu+q-r+1}[y^{\mu+q-r+1+1/2}\\
&\times(y^{-1}d/dy)\big\{y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)(x^{-1}d/dx)^r a(x,y)\big\}]\\
&=\sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}x^r\int_0^\infty (xy)^{1/2}J_{\mu+q-r+2}(xy)N_{\mu+q-r+1}[y^{\mu+q-r+1+1/2}
\end{aligned}
$$

$$\times(y^{-1}d/dy)\{y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)(x^{-1}d/dx)^r a(x,y)\}]$$
$$=\sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}\int_0^\infty x^{r+1/2}y^{\mu+q-r+2+1}$$
$$\times[(y^{-1}d/dy)^2\{y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)(x^{-1}d/dx)^r a(x,y)\}]J_{\mu+q-r+2}(xy)dy.$$

In general, we have

$$(-x)^k(N_{\mu+q-1}\ldots N_\mu\Phi(x))=\sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}\int_0^\infty x^{r+1/2}y^{\mu+q-r+k+1}$$
$$\times[(y^{-1}d/dy)^k\{y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)(x^{-1}d/dx)^r a(x,y)\}]J_{\mu+q-r+k}(xy)dy$$
$$=\sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}\int_0^\infty x^{r+1/2}y^{\mu+q-r+k+1}\sum_{\nu=0}^{k}\binom{k}{\nu}(y^{-1}d/dy)^\nu$$
$$\times(x^{-1}d/dx)^r a(x,y)(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)J_{\mu+q-r+k}(xy)dy. \quad (42)$$

Now, multiplying both sides in (38) by $(-x)^k$, we get

$$(-x)^k(N_{\mu+q-1}\ldots N_\mu\Phi(x))=(-1)^k x^{\mu+k+q+1/2}(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x). \quad (43)$$

Comparing equations (42) and (43), we have

$$(-1)^k x^{\mu+k+q+1/2}(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)=\sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}\int_0^\infty x^{r+1/2}$$
$$\times y^{\mu+q-r+k+1}\sum_{\nu=0}^{k}\binom{k}{\nu}(y^{-1}d/dy)^\nu(x^{-1}d/dx)^r a(x,y)$$
$$\times(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)J_{\mu+q-r+k}(xy)dy.$$

Therefore,

$$(-1)^k x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)=\sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}\int_0^\infty x^{-(\mu+q-r)}y^{\mu+q-r+k+1}$$
$$\times\sum_{\nu=0}^{k}\binom{k}{\nu}(y^{-1}d/dy)^\nu(x^{-1}d/dx)^r a(x,y)(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}$$
$$\times(h_\mu\tau_z(\phi\#\psi))(y)J_{\mu+q-r+k}(xy)dy.$$

Thus

$$\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)\right|$$
$$\leq\sum_{r=0}^{q}\binom{q}{r}\int_0^\infty y^{2(\mu+q-r)+k+1}\sum_{\nu=0}^{k}\binom{k}{\nu}\left|(x^{-1}d/dx)^r(y^{-1}d/dy)^\nu a(x,y)\right|$$
$$\times\left|(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)\right|\left|(xy)^{-(\mu+q-r)}J_{\mu+q-r+k}(xy)\right|dy.$$

Using inequality (36) and let $E > 0$, the right-hand side assumes the form

$$\sum_{r=0}^{q}\binom{q}{r}\int_0^\infty y^{2(\mu+q-r)+k+1}\sum_{\nu=0}^{k}\binom{k}{\nu}L_m(C+\delta)^r c_r(D+\eta)^\nu d_\nu(1+y)^{m-\nu}$$
$$\times\left|(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)\right|\frac{2^{-(\mu+q-r)}E}{\Gamma(\mu+q-r+1)}dy. \tag{44}$$

If we assume that p is a positive integer such that $p \geq m$ and $s > 2\mu+1$, then the last term can be estimated by

$$\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)\right| \leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)}\sum_{r=0}^{q}\sum_{\nu=0}^{k}\binom{q}{r}\binom{k}{\nu}(C+\delta)^r c_r(D+\eta)^\nu d_\nu$$
$$\times\int_0^\infty y^{2(\mu+q-r)+k+1}(1+y)^{m-\nu+s}\left|(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)\right|\frac{dy}{(1+y)^s}$$

$$\leq \frac{2^{-\mu}EL_m}{\Gamma\mu+1}\sum_{r=0}^{q}\sum_{\nu=0}^{k}\binom{q}{r}\binom{k}{\nu}(C+\delta)^r c_r(D+\eta)^\nu d_\nu \sup_{y\in I}\left[(1+y)^{p+s}y^{2(q-r)+k}\right.$$
$$\left.\times\left|(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)\right|\right]\int_0^\infty\frac{y^{2(\mu+1/2)}}{(1+y)^s}dy$$
$$\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Phi(x)\right| \leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)}\sum_{r=0}^{q}\sum_{\nu=0}^{k}\binom{q}{r}\binom{k}{\nu}(C+\delta)^r c_r(D+\eta)^\nu d_\nu$$
$$\times\sum_{n=0}^{p+s}\binom{p+s}{n}\sup_{y\in I}\left[y^n y^{2(q-r)+k}\left|(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)\right|\right]$$
$$\leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)}\sum_{r=0}^{q}\sum_{\nu=0}^{k}\sum_{n=0}^{p+s}\binom{q}{r}\binom{k}{\nu}\binom{p+s}{n}(C+\delta)^r c_r(D+\eta)^\nu d_\nu$$
$$\times\sup_{y\in I}\left|y^{n+2(q-r)+k}(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)\right|$$
$$\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}(h_{\mu,a}\tau_z(\phi\#\psi))(x)\right| \leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)}\sum_{r=0}^{q}\sum_{\nu=0}^{k}\sum_{n=0}^{p+s}\binom{q}{r}\binom{k}{\nu}\binom{p+s}{n}$$
$$\times(C+\delta)^r c_r(D+\eta)^\nu d_\nu \sup_{y\in I}\left|y^{n+2(q-r)+k}(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}(h_\mu\tau_z(\phi\#\psi))(y)\right|. \tag{45}$$

Using Theorem (2.4)(i) in (45), we have

$$\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}(h_{\mu,a}\tau_y(\phi\#\psi))(x)\right| \leq \frac{2^{-\mu}EL_m}{\Gamma(\mu+1)}\sum_{r=0}^{q}\sum_{\nu=0}^{k}\sum_{n=0}^{p+s}\binom{q}{r}\binom{k}{\nu}\binom{p+s}{n}$$
$$\times(C+\delta)^r c_{q-(q-r)}(D+\eta)^\nu d_{k-(k-\nu)}\left\|h_\mu\tau_z(\phi\#\psi)\right\|^{\mu}_{n-2(q-r)+k}(B_3+\rho)^{k-\nu}a^2_{k-\nu}.$$

Using the inequality (27), then right-hand side of the above inequality is bounded by

$$\frac{2^{-\mu}EL_m}{\Gamma(\mu+1)}\sum_{r=0}^{q}\sum_{\nu=0}^{k}\sum_{n=0}^{p+s}\binom{q}{r}\binom{k}{\nu}\binom{p+s}{n}(C+\delta)^r c_q(c_0/c_1)^{q-r}(D+\eta)^{\nu}a_k^*(a_0^*/a_1^*)$$
$$\times(a_{k-\nu}^*a_{\nu}^*)a_{k-\nu}^*(B_3+\rho)^{k-\nu}\left\|h_\mu\tau_z(\phi\#\psi)\right\|_{n-2(q-r)+k}^{\mu},$$

where $a_\nu^*=\max_{\nu\in\mathbb{N}_0}(a_\nu,d_\nu)$ and using inequalities (25) and (27), we have

$$\leq\frac{2^{-\mu}EL_m}{\Gamma(\mu+1)}\sum_{r=0}^{q}\sum_{\nu=0}^{k}\sum_{n=0}^{p+s}\binom{q}{r}\binom{k}{\nu}\binom{p+s}{n}(C+\delta)^r(c_0/c_1)^{q-r}c_q(D+\eta)^{\nu}(a_0^*/a_1^*)^{\nu}a_k^*$$
$$\times(a_0^*a_k^*)(B_3+\rho)^{k-\nu}\left\|h_\mu\tau_z(\phi\#\psi)\right\|_{n-2(q-r)+k}^{\mu}$$

$$\leq\frac{2^{-\mu}EL_m a_0^*}{\Gamma(\mu+1)}\sum_{r=0}^{q}(C+\delta)^r(c_0/c_1)^{q-r}c_q\sum_{\nu=0}^{k}\binom{k}{\nu}((a_0^*/a_1^*)(D+\eta))^{\nu}(B_3+\rho)^{k-\nu}a_k^{*2}$$
$$\times\sum_{n=0}^{p+s}\binom{p+s}{n}\max_{0\leq n\leq p+s}\max_{0\leq r\leq q}\left\|h_\mu\tau_z(\phi\#\psi)\right\|_{n-2(q-r)+k}^{\mu}$$
$$\leq Q_q^{\mu}(A_3+\acute{\rho})^k a_k^{*2}.$$

Therefore,

$$\|\Phi\|_q^{\mu}=\sup_{k\in\mathbb{N}_0}\sup_{x\in I}\frac{\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}h_{\mu,a}\tau_z(\phi\#\psi)(x)\right|}{(A_3+\acute{\rho})^k a_k^{*2}}$$
$$\leq L((c_0/c_1)+(C+\delta))^q c_q\max_{0\leq n\leq p+s}\max_{0\leq r\leq q}\left\|h_\mu\tau_z(\phi\#\psi)\right\|_{n-2(q-r)+k}^{\mu},$$

where $\acute{\rho}>0$ and L is a positive constant.

Hence $h_{\mu,a}\tau_z(\phi\#\psi)(x)\in H_{\mu,a_k^{*2},A_3}$.

REMARK:1 In Theorem (2.5), we may also choose $\phi,\psi\in\widetilde{H}_{\mu}^{b_q,B}$ (or $H_{\mu,a_k,A}^{b_q,B}$) then the p.d.o. $h_{\mu,a}\tau_z(\phi\#\psi)\in H_{\mu}^{b_q^{*2},B}$ (or $H_{\mu,a_k^{*3}b_k^*,A_2}^{a_q^{*2}b_q^{*2},B_4}$) where $B_4=\left[(b_0^*a_0^{*2}/b_1^*a_1^{*2})C+H^6A_1^2\right]$, $A_2=(a_0^*/a_1^*)D+B_1$, $A_1=AB(R^*)^2$ and $B_1=A^2(R^*)^6$.

3 Hankel potential involving Hankel translation and Hankel convolution of the spaces of type H_μ

This section investigates the Hankel potential involving Hankel translation and Hankel convolution on the spaces $H_{\mu,a_k,A}$, $H_{\mu}^{b_q,B}$ and $H_{\mu,a_k,A}^{b_q,B}$.

From [4], we have the following interesting result.

Lemma 3.1 *If $s \in \mathbb{R}$, then the derivatives of $a(y) = (1+y^2)^{-s/2}$ satisfy*

$$\left|(y^{-1}d/dy)^\beta a(y)\right| \leq L_s(D+\eta)^\beta d_\beta(1+y)^{-s-\beta},\ \beta = 0,1,2,..., \tag{46}$$

where $\{d_\beta\}$ is a certain sequence of positive real numbers satisfying the conditions (22)–(29) and $\eta \geq |s|/2$.

Proof: Let $a(y) = (1+y^2)^{-s/2}$;
then

$$(y^{-1}d/dy)a(y) = (-1)2(s/2)(1+y^2)^{-(s/2)-1}.$$

In general, we get

$$(y^{-1}d/dy)^\beta a(y) = (-1)^\beta 2^\beta (s/2)((s/2)+1)\cdots((s/2)+\beta-1)(1+y^2)^{-(s/2)-\beta};$$

so that

$$\begin{aligned}\left|(y^{-1}d/dy)^\beta a(y)\right| &\leq 2^\beta(|s|/2)((|s|/2)+1)\cdots((|s|/2)+\beta-1)(1+y^2)^{-(s/2)-\beta},\\ &= 2^\beta\frac{\Gamma((|s|/2)+\beta)}{\Gamma(|s|/2)}(1+y^2)^{-(s/2)-\beta}.\end{aligned} \tag{47}$$

It can be easily shown that

$$(1+y^2)^{-(|s|/2)-\beta} \leq 2^{2\beta}A_s(1+y)^{-s-\beta}. \tag{48}$$

From [5], we know that

$$\Gamma(\eta+\beta) \leq B(D+\eta)^\beta d_\beta. \tag{49}$$

Using results (47), (48) and (49), for $\eta \geq |s|/2$, we get

$$\begin{aligned}\left|(y^{-1}d/dy)^\beta a(y)\right| &\leq B(D+\eta)^\beta d_\beta\frac{2^{3\beta}A_s}{\Gamma(|s|/2)}(1+y)^{-s-\beta}\\ &\leq L_s(D+\eta)^\beta d_\beta(1+y)^{-s-\beta}.\end{aligned}$$

It follows that the $a(y) \in S^{-s}_{d_\beta}$. The corresponding pseudo-differential operator denoted by $h_{\mu,a}$ will be called the Hankel potential of order s and it is denoted by h^s_μ and defined by (4).

Theorem 3.2 *Let $\{a_k\}$ and $\{d_k\}, k \in \mathbb{N}_0$, satisfy condition (27), for each fixed $z, 0 < z < z_0$ and $\mu \geq -1/2$ and let the symbol $a(y) = (1+y^2)^{-s/2}$ satisfy (46), then the Hankel potential $(\phi,\psi) \longmapsto h^s_\mu\tau_z(\phi\#\psi)$ is continuous linear mapping from $H_{\mu,a_k,A} \times H_{\mu,a_k,A}$ into $\widetilde{H}_{\mu,a_k^{*2},A_6'}$, where $A_6' = (a_0^*/a_1^*)D + B_3$, B_3 as Theorem (2.4) and $a_k^* = \max_{k\in\mathbb{N}_0}(a_k, d_k)$.*

Proof: Let $(\phi,\psi)\in H_{\mu,a_k,A}\times H_{\mu,a_k,A}$, then by Theorem (2.3)(i) $h_\mu(\phi\#\psi)\in H_\mu^{a_q^2,B_1}$ and by Theorem (2.4)(i) $h_\mu\tau_z(\phi\#\psi)\in H_\mu^{a_q^2,B_3}$.

Now assume that

$$\Theta(x)=h_\mu^s\tau_z(\phi\#\psi)(x)=\int_0^\infty (xy)^{1/2}J_\mu(xy)a(y)h_\mu\tau_z(\phi\#\psi)(y)dy.$$

Now proceed as proof of Theorem (2.5), we have the inequality (44) is as given below

$$\begin{aligned}&\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Theta(x)\right|\le\frac{2^{-\mu}EL_s}{\Gamma(\mu+1)}\int_0^\infty L_s(D+\eta)^\nu d_\nu(1+y)^{-s-\nu}\\&\times\left|y^{2(\mu+q)+k+1}(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}h_\mu\tau_z(\phi\#\psi)(y)\right|dy\\&\le\frac{2^{-\mu}EL_s}{\Gamma(\mu+1)}\sum_{\nu=0}^{k}\binom{k}{\nu}(D+\eta)^\nu d_\nu\sup_{y\in I}\left|y^{2q+k}(1+y)^{t-m}(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}\right.\\&\times\left.h_\mu\tau_z(\phi\#\psi)(y)\right|\int_0^\infty\frac{y^{2(\mu+1/2)}}{(1+y)^t}dy.\end{aligned}$$

Let m be an integer less then or equal to s. Then $(1+y)^{t-s-\nu}\le(1+y)^{t-m-\nu}$ $\le(1+y)^{t-m}$, we have

$$\begin{aligned}&\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Theta(x)\right|\le\frac{2^{-\mu}EL_s}{\Gamma(\mu+1)}\sum_{\nu=0}^{k}\sum_{n=0}^{t-m}\binom{k}{\nu}\binom{t-m}{n}(D+\eta)^\nu d_\nu\\&\times\sup_{y\in I}\left|y^{n+2q+k}(y^{-1}d/dy)^{k-\nu}y^{-\mu-1/2}h_\mu\tau_z(\phi\#\psi)(y)\right|.\end{aligned}$$

Using Theorem (2.4)(i), the right hand-side assumes the form

$$\begin{aligned}&\frac{2^{-\mu}EL_s}{\Gamma(\mu+1)}\sum_{\nu=0}^{k}\sum_{n=0}^{t-m}\binom{k}{\nu}\binom{t-m}{n}(D+\eta)^\nu d_\nu\left\|h_\mu\tau_z(\phi\#\psi)\right\|_{n+2q+k}^{\mu}(B_3+\rho)^{k-\nu}a_{k-\nu}^2\\&\le\frac{2^{-\mu}EL_s}{\Gamma(\mu+1)}\sum_{\nu=0}^{k}\sum_{n=0}^{t-m}\binom{k}{\nu}\binom{t-m}{n}(D+\eta)^\nu(a_{k-\nu}^*a_\nu^*)a_{k-\nu}^*(B_3+\rho)^{k-\nu}\\&\times\left\|h_\mu\tau_z(\phi\#\psi)\right\|_{n+2q+k}^{\mu},\end{aligned}$$

where $a_\nu^* = \max_{\nu\in\mathbb{N}_0}(a_\nu, d_\nu)$ and using inequalities (25) and (27), we have

$$\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Theta(x)\right| \leq \frac{2^{-\mu}EL_s}{\Gamma(\mu+1)}\sum_{\nu=0}^{k}\sum_{n=0}^{t-m}\binom{k}{\nu}\binom{t-m}{n}(D+\eta)^\nu$$
$$\times(a_0^*/a_1^*)^\nu a_k^*(a_0^* a_k^*)(B_3+\rho)^{k-\nu}\left\|h_\mu\tau_z(\phi\#\psi)\right\|_{n+2q+k}^{\mu}$$
$$\leq \frac{2^{-\mu}EL_s a_0^*}{\Gamma(\mu+1)}\sum_{\nu=0}^{k}\binom{k}{\nu}((a_0^*/a_1^*)(D+\eta))^\nu(B_3+\rho)^{k-\nu}a_k^{*2}$$
$$\times\sum_{n=0}^{t-m}\binom{t-m}{n}\max_{0\leq n\leq t}\left\|h_\mu\tau_z(\phi\#\psi)\right\|_{n+2q+k}^{\mu}$$
$$\leq Q_q^\mu(A_6'+\acute{\rho})^k a_k^{*2}.$$

Therefore,

$$\|\Theta\|_q^\mu = \sup_{k\in\mathbb{N}_0}\sup_{x\in I}\frac{\left|x^k(x^{-1}d/dx)^q x^{-\mu-1/2}\Theta(x)\right|}{(A_6'+\acute{\rho})^k a_k^{*2}}$$
$$\leq L\max_{0\leq n\leq t}\left\|h_\mu\tau_z(\phi\#\psi)\right\|_{n+2q+k}^{\mu},$$

where $A_6' = (a_0^*/a_1^*)D + B_3, B_3$ as Theorem (2.4), $\acute{\rho} > 0$ and L is a constant.

Hence $h_{\mu,a}\tau_z(\phi\#\psi)(x) \in \widetilde{H}_{\mu,a_k^{*2},A_6'}$.

REMARK: In Theorem (3.2) we may also choose $\phi,\psi \in \widetilde{H}_\mu^{b_q,B}(or H_{\mu,a_k,A}^{b_q,B})$ then the Hankel potential $h_\mu^s\tau_z(\phi\#\psi) \in \widetilde{H}_\mu^{b_q^2,B_6'}(or H_{\mu,a_k^{*3}b_k,A_6'}^{a_q^{*2}b_q^2,B_6})$, where $B_6' = H_2^6B^2, A_6 = ((a_0^*/a_1^*)D+B_3)A_1, B_6 = H^6A_1^2$ and A_1, B_3 as Theorem (2.4).

ACKNOWLEDGEMENTS. This work is supported by Indian School of Mines, under Grant No. MRP/Acad/05/2008.

References

[1] J. J. Betancor and I. Marrero, The Hankel convolution and Zemanian spaces β_μ and β'_μ , *Math Nachr*, **160** (1993), 277-298.

[2] I. M. Gel'fand and G. E. Shilov, Generalized functions, *Academic Press,* **2** 1968.

[3] W. Y. K. Lee, On spaces of type H_μ and their Hankel transformation, *SIAM J. Math. Anal.* **5**(2) (1974), 336-348.

[4] R. S. Pathak and A. Prasad, Continuity of pseudo-differential operators associated with Bessel operator in some Gevrey spaces, *Applicable Analysis,* **81** (2002), 637-662.

[5] R. S. Pathak and K. K. Shrestha, Hankel convolution on some ultradifferentiable function spaces, *Function Spaces and Applications* (Eds. D. E. Edmunds et al), Narosa Publishing House, New Delhi (2000), 232-243.

[6] L. Rodino, *Linear partial differential operators in Gevrey spaces,* World Scientific, Singapore 1993.

[7] M. W. Wong, *An introduction to pseudo-differential operators,* World Scientific, Singapore 1999.

[8] A. H. Zemanian, *Generalized integral transformations,* Interscience Publishers, New York, 1968.

Received: March, 2010