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Strong Convergence of Two Iterative Algorithms for a Countable Family of Nonexpansive Mappings in Hilbert Spaces

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Abstract

In this paper, we study two iterative algorithms for a countable family of nonexpansive mappings in Hilbert Spaces.We prove that the proposed algorithms converge strongly to a fixed point of nonexpansive mappings $\{T_n\}$.The results of this paper extend and improve the results of Yonghong Yoa, et al. [14].

Mathematics Subject Classification: 47H05, 47H10

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1 Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. And let C be a nonempty closed convex subset of H. A mapping T of C into itself is said to be nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for each $x, y \in C$. We denote by $F(T)$ the set of fixed points of T.

In 2003, for finding an element of $F(S) \cap VI(A,C)$, Takahashi and Toyoda [10] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) SPc(x_n - \lambda_n Ax_n)$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in \mathcal{C}$,$\{\alpha_n\}$is a sequence in $(0,1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$.

Let A is a strongly positive bounded linear operator on H. That is, there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \text{ for all } x \in H.$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbeert space H.

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle\ -\langle x,b\rangle$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H.

Recently, Xu [12] proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0\in H$ chosen arbitrarily:

$$x_{n+1}=(I-\alpha_nA)Tx_n+\alpha_nu,\ n\geq 0, \tag{1.1}$$

converges strongly to the unique solution of the minimization problem provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

On the other hand, Aoyama, et al.,[1] introduce a Halpern type iterative sequence for finding a common fixed point of a countable family of nonexpansive mappings. Let $x_1=x\in\mathcal{C}$ and

$$x_{n+1}=\alpha_nx+(1-\alpha_n)T_nx_n \tag{1.2}$$

for all $n\in\mathbb{N}$, where C is a nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some condition. They proved that $\{x_n\}$ defined by (1.2) converges strongly to a common fixed point of $\{T_n\}$.

Very recently, Yonghong Yoa,et al.[14] introduced two iterative algorithms defined by, for given $x_0\in C$ arbitrarily and let the sequence $\{x_n\},n\geq 0$ be generated by

$$\begin{cases} y_n=P_C[(1-\alpha_n)x_n], \\ x_{n+1}=(1-\beta_n)x_n+\beta_nTy_n, \end{cases} \tag{1.3}$$

and they prove that the algorithms strongly converge to a fixed point of nonexpansive mappings T.

The research in this field, iterative algorithms for finding fixed points of nonlinear mappings, is important and find applications in a variety of applied areas of inverse problem, partial differential equations,image recovery and signal processing, see [2] [3] [4] [6].

In this paper motivate by result of Yonghong Yoa,et al.,and the ongoing research in this field, we introduced the two iterative algorithms in Hilbert space defined by, for given $x_0\in C$ arbitrarily and let the sequence $\{x_n\},n\geq 0$ be generated by

$$\begin{cases} y_n=P_C[(1-\alpha_n)x_n], \\ x_{n+1}=(1-\beta_n)x_n+\beta_nT_ny_n, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real in [0,1] and $\{T_n\}$ is a sequence of nonexpansive mappings with some conditions. Then we prove that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a fixed point of $\{T_n\}$. The result of this result extends and improves the corresponding results of Yonghong Yoa,et al.,[14].

2 Preliminary Notes

Let H be a real Hilbert space and let C be a closed convex subset of H.Then, for any $x \in H$, there exists a unique nearest point $u \in C$ such that

$$\|x - u\| \leq \|x - y\|, \forall y \in C.$$

We denote u by $P_C(x)$, where P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive. Furthermore, for $x \in H$ and $u \in C$,

$$u = P_C(x) \Leftrightarrow \langle x - u, u - y\rangle \geq 0, \forall y \in C$$

Lemma 2.1. ([12]) *Let C be a nonempty closed convex of a real Hilbert space H. Let $T : C \longrightarrow C$ be a nonexpansive mapping. Then $I - T$ is demi-closed at zero, i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$*

Lemma 2.2. ([8]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\longrightarrow\infty} \beta_n \leqslant \limsup_{n\longrightarrow\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n+\beta_n x_n$ for all integer $n \geq 0$ and $\limsup_{n\longrightarrow\infty}(\|y_{n+1}- y_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. Then $\lim_{n\longrightarrow\infty} \|y_n - x_n\| = 0$.*

Lemma 2.3. ([12]).*Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n$, for all $n \geq 0$ where $\{\gamma_n\}$ be a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathcal{R}$ such that*

(i) $\sum_{n=10}^{\infty} \gamma_n = \infty$,

(ii) $\lim sup_{n\longrightarrow\infty}\delta_n \leq 0$ *or* $\sum_{n=10}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.4. *.Let H be a real Hilbert space.Then for all$x, y \in H$,the following hold;*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle y, x + y\rangle$,

(ii) $\|x + y\|^2 = \|x\|^2 + 2\langle y, x\rangle$

Throughout the rest of this paper, Let each $t \in (0,1)$, we consider the following mapping T_t given by

$$T_t x = TP_C[(1-t)x], \quad \forall x \in C.$$

It is easy to check that $\|T_t x - T_t y\| \leq (1-t)\|x-y\|$ which implies that T_t is a contraction. Using the Banach contraction principle, there exists a unique fixed point x_t of T_t in C, i.e.,

$$x_t = TP_C[(1-t)x_t]. \tag{2.1}$$

Lemma 2.5. ([14]). *Let C be a nonempty bounded closed convex subset of Hilbert space H. Let $T : C \longrightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For each $t \in (0,1)$, let the net $\{x_t\}$ be generated by (2.1). Then, as $t \to 0$, the net $\{x_t\}$ converges strongly to a fixed point of T.*

Lemma 2.6. ([11]). *Let C be a nonempty bounded closed convex subset of Hilbert space H and $\{T_n\}$ be a sequence of mappings of C into itself. Suppose that*

$$\lim_{k,l\to\infty} \rho_l^k = 0, \tag{2.2}$$

where $\rho_l^k = \sup\{\|T_k z - T_l z\| : z \in C\} < \infty$, for all $k, l \in \mathbb{N}$. Then for each $x \in C$, $\{T_n x\}$ converges strongly to some point of C. Moreover, let T be a mapping from C into itself defined by $Tx = \lim_{n\to\infty} T_n x$, for all $x \in C$. Then $\limsup_{n\to\infty}\{\|Tz - T_n z\| : z \in C\} = 0$

3 Main results

In this section, we prove the strong convergence theorems for a countable family of nonexpansive mappings in a real Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a sequence of nonexpansive mapping of H into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $(0,1)$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$, $n \geq 0$. be generated by (1.4). Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=10}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=10}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=10}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(iii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

Suppose that for any $C of H$, the sequence $\{T_n\}$ satisfied condition (2.2) in lemma (2.6). Let T be a mapping of H into itself defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in H$. If $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ then $\{x_n\}$ converges strongly to a fixed point z in $F(T)$.

Proof. First, we observed that $\{x_n\}$ is bounded.
Indeed, pick any $p \in \bigcap_{n=1}^{\infty} F(T_n) = F(T)$ to obtain,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\beta_n)x_n + \beta_n T_n y_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|T_n y_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|y_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|(1-\alpha_n)x_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n[(1-\alpha_n)\|x_n - p\| + \alpha_n\|p\|] \\
&\leq (1-\alpha_n\beta_n)\|x_n - p\| + \alpha_n\beta_n\|p\| \\
&\leq max\{\|x_n - p\|, \|p\|\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded and so is $\{T_n x_n\}$.

Let $B = \{y \in H : \|y - p| \ \leq K\}$ where $K = max\{\|x_n - p\|, \|p\|\}, \quad n \geq 0$. Clearly, B is bounded closed convex subset of H, $T(B) \subseteq B$, $\{x_n\} \subseteq B$, and $\{T_n x_n\} \subseteq B$.
Set $z_n = T_n y_n, \quad n \geq 0$. It follows that

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \|T_{n+1}y_{n+1} - T_{n+1}y_n\| + \|T_{n+1}y_n - T_n y_n\| \\
&\leq \|(1-\alpha_{n+1})x_{n+1} - (1-\alpha_n)x_n\| + \|T_{n+1}y_n - T_n y_n\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\| \\
&\quad + \|T_{n+1}(1-\alpha_{n+1})x_{n+1} - T_n(1-\alpha_n)x_n\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\| + \sup_{s\in B}\|T_{n+1}s - T_n s\|.
\end{aligned}$$

Then, we obtained

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| = \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\| + \sup_{s\in B}\|T_{n+1}s - T_n s\|.$$

By our assumptions, we get

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

This together with lemma (2.2) imply that $\lim_{n\to\infty}\|z_n - x_n\| = 0$.
Therefore,

$$\begin{aligned}
\lim_{n\to\infty}\|x_{n+1} - x_n\| &= \lim_{n\to\infty}\|(1-\beta_n)x_n + \beta_n z_n - x_n\| \\
&= \lim_{n\to\infty}\beta_n\|z_n - x_n\| = 0
\end{aligned}$$

Now, we observe that

$$\begin{aligned}\|x_n - T_n x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\| \\ &\le \|x_n - x_{n+1}\| + \|(1-\beta_n)x_n - \beta_n T_n y_n - T_n x_n\| \\ &\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T_n x_n\| + \beta_n\|y_n - x_n\| \\ &\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T_n x_n\| + \beta_n\|(1-\alpha_n)x_n - x_n\| \\ &\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T_n x_n\| + \alpha_n\|x_n\|\end{aligned}$$

That is $\|x_n - T_n x_n\| \le \frac{1}{\beta_n}\{\|x_n - x_{n+1}\| + \alpha_n\|x_n\|\} \to 0 \;\; as \;\; n \to \infty$.
Let the net $\{x_t\}$ defined by (2.1). By lemma (2.5), we have $x_t \to z$ as $t \to 0$

Next,we prove that $\limsup_{n\to\infty}\langle z, z - x_n\rangle \le 0$. Indeed, we calculate

$$\begin{aligned}\|x_t - x_n\|^2 &= \|(x_t - T_n x_n) + (T_n x_n - x_n)\|^2 \\ &= \|x_t - T_n x_n\|^2 + 2\langle x_t - T_n x_n, T_n x_n - x_n\rangle + \|T_n x_n - x_n\|^2 \\ &\le \|x_t - T_n x_n\|^2 + 2\langle x_t - x_n, T_n x_n - x_n\rangle \\ &\quad -2\langle T_n x_n - x_n, T_n x_n - x_n\rangle + \|T_n x_n - x_n\|^2 \\ &\le \|T_n P_C[(1-t)x_t - T_n x_n\|^2 + 2\|x_t - x_n\|\|T_n x_n - x_n\| \\ &\quad -\|T_n x_n - x_n\|^2 \\ &\le \|(1-t)x_t - x_n\|^2 + 2\|x_t - x_n\|\|T_n x_n - x_n\| \\ &\le \|x_t - x_n\|^2 - 2t\langle x_t, x_t - x_n\rangle + t^2\|x_t\|^2 \\ &\quad +2\|x_t - x_n\|\|T_n x_n - x_n\| \\ &\le \|x_t - x_n\|^2 - 2t\langle x_t, x_t - x_n\rangle + t^2 M + M\|T_n x_n - x_n\|\end{aligned}$$

where $M > 0$ such that $\sup\{\|x_t\|^2, 2\|x_t - x_n\|, t \in (0,1), n \ge 1\} \le M$.
It follows that $\langle x_t, x_t - x_n\rangle \le \frac{t}{2}M + \frac{M}{2t}\|x_n - T_n x_n\|$.
Therefore,

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t, x_t - x_n\rangle \le 0. \tag{3.1}$$

We note that

$$\begin{aligned}\langle z, z - x_n\rangle &= \langle z, z - x_t\rangle + \langle z - x_t, x_t - x_n\rangle + \langle x_t, x_t - x_n\rangle \\ &\le \langle z, z - x_t\rangle + \|z - x_t\|\|x_t - x_n\| + \langle x_t, x_t - x_n\rangle \\ &\le \langle z, z - x_t\rangle + \|z - x_t\|M + \langle x_t, x_t - x_n\rangle.\end{aligned}$$

This together with $x_t \to z$ and (3.1) implies that

$$\limsup_{n\to\infty}\langle z, z - x_n\rangle \le 0$$

Finally, we show that $x_n \to z$. From (1.4), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|(1-\beta_n)x_n + \beta_n T_n y_n - z\|^2 \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\|T_n y_n - z\|^2 \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\|y_n - z\|^2 \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\|(1-\alpha_n)(x_n - z) - \alpha_n z\|^2 \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n[(1-\alpha_n)\|x_n - z\|^2 \\
&\quad -2\alpha_n(1-\alpha_n)\langle z, x_n - z\rangle + \alpha_n^2\|z\|^2] \\
&\le (1-\alpha_n\beta_n)\|x_n - z\|^2 + \alpha_n\beta_n[2(1-\alpha_n)\langle z, z - x_n\rangle + \frac{\alpha_n}{\beta_n}\|z\|^2].
\end{aligned}$$

By lemma (2.3), we obtained $x_n \to z$ as $n \to \infty$. This completes the proof. □

Setting $T_n \equiv T$ in Theorem 3.1, we have the following result.

Corollary 3.2. *[14, Theorem 3.2] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty. Let $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $(0,1)$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$, $n \ge 0$. be generated by (1.3). Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=10}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\longrightarrow\infty} \beta_n \le \limsup_{n\longrightarrow\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

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