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A Representation of a Class of Heyting Algebras by Fractions

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Abstract

In this paper, we solve an open problem in an special case. The problem is to give a characterization for Heyting algebras by means of fractions. Here, we give a representation for a class of Heyting algebras by means of fractions. Fractions on a bounded distributive lattice is a new algebraic structure, which was recently studied by the authors.

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1 Introduction and preliminaries

A characterization of distributive lattices as a ring of sets and Boolean lattices as a field of sets are well known [4, 9]. A characterization of pseudo-Boolean lattices (or a Heyting algebra [4] and [3]) is given in [1] by means of residuated mappings (see [1], Theorem 7.9). In [7] we have constructed a new algebraic structure, as fractions on a lattice, and by means of it we characterized finite Heyting algebras. But giving a characterization for Heyting algebras in general, was left as an open problem. In this paper we give a representation of a class of Heyting algebras by means of fractions which was studied in [7]. Moreover, pseudo-Boolean lattices (or Heyting algebras) have main role in some optimization problems over lattices. The optimization problem over distributive lattices was studied in [5], and in particular case this problem was studied over pseudo-Boolean lattices in [6] (also see [8]).

We give some definitions and theorems which we need in the sequel in order to give our representation. For more details see the references.

Definition 1.1. Let H be a non-empty set. A subset R of $P(H)$ is called a ring of sets if the union $X \cup Y$ and the intersection $X \cap Y$ belongs to R for all X and Y in R. A ring of sets R is called a field of sets if $X^c \in R$, for all $X \in R$, where $X^c = H \backslash X$. Note that $(R, \subseteq)$ is a lattice, where $\subseteq$ is the inclusion of sets.

Example 1.2. Let $H = \{2, 3, 4, 9\}$. We can consider a ring of sets R as follows:

$$R = \{\phi, \{2\}, \{3\}, \{2,3\}, \{2,4\}, \{3,9\}, \{2,3,4\}, \{2,3,9\}, \{2,3,4,9\}\}.$$

Definition 1.3. A bounded lattice $(L, \leq)$ is called a pseudo-Boolean if for all $a, b \in L$, there exists $c \in L$ such that

$$a \wedge x \leq b \Leftrightarrow x \leq c \qquad \forall x \in L.$$

If such element c exists then, it is unique and will be denoted by $b : a$.

Definition 1.4. ([3]) A Heyting algebra is an algebraic structure $A = (A; \vee, 0, \wedge, 1, \rightarrow)$ such that $(A; \vee, 0, \wedge, 1)$ is a bounded lattice, and $\rightarrow$ gives the residual of $\wedge$:

$$a \wedge x \leq b \Leftrightarrow x \leq a \rightarrow b.$$

Remark 1.5. We see that a pseudo-Boolean lattice is exactly a Heyting algebra where, $b : a$ is $a \rightarrow b$.

Remark 1.6. ([9]) (i) Every finite distributive lattice is pseudo-Boolean.
(ii) Every Boolean lattice is pseudo-Boolean.
(iii) In general, a pseudo-Boolean lattice may not be Boolean. For example, consider a bounded linearly ordered set $(B, \leq)$, where $a \wedge b = min(a, b)$ and $a \vee b = max(a, b)$ and for all $a, b \in B$,

$$b : a = \begin{cases} 1 & if \quad b \geq a \\ b & if \quad b < a \end{cases}$$

B is not Boolean, since for any a, $0 < a < 1$, we have $a \vee (0 : a) = a \vee 0 = a < 1$.

Theorem 1.7. ([9], Proposition 1.17.) Let $(L, \leq)$ be a lattice. If L is a pseudo-Boolean lattice, then it is distributive.

Definition 1.8. ([4]) Let $(L, \leq)$ be a lattice.
(i) A sublattice I of L is an ideal if and only if $i \in I$ and $a \in L$ implies $a \wedge i \in I$.
(ii) A proper ideal I of L is prime if and only if $a, b \in L$ and $a \wedge b \in I$ imply that $a \in I$ or $b \in I$.

Definition 1.9. ([2]) A lattice $(L, \leq)$ is called infinitely distributive if

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i), \qquad (1)$$

and

$$a \vee (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \vee b_i). \qquad (2)$$

Equality (1) is called join infinite distributive identity or JID. In the same way, (2) is called meet infinite distributive identity or MID (see [4]).

Note that, JID and MID may not hold in every complete distributive lattice. Also they may not imply each other.

Example 1.10. ([2]) Let $(L, \subseteq)$ be the complete lattice of all closed subsets of the plane. Let c denote the circle $x^2 + y^2 = 1$ and d_k denote the disc $x^2 + y^2 \leq 1 - k^{-2}$, then $c \wedge (\bigvee_{k=1}^{\infty} d_k) = c$ and $\bigvee_{k=1}^{\infty} (c \wedge d_k)$ is the empty set. Therefore, JID does not hold. On the other hand, MID holds, since in this case $\vee$ and $\wedge$ coincide with the set theoretic operations $\cup$ and $\cap$, respectively.

Theorem 1.11. ([4], Corollary II.1.17.) Let L be a distributive lattice, $a, b \in L$ and $a \neq b$. Then, there is a prime ideal P such that $a \in P$ or $b \in P$.

Corollary 1.12. ([4], Corollary II.1.18.) Every ideal I of a distributive lattice is the intersection of all prime ideals containing it.

The following theorem characterize any distributive lattice.

Theorem 1.13. ([4], Theorem II.1.19) A lattice is distributive if and only if it is isomorphic to a ring of sets.

Definition 1.14. ([4]) A lattice $(L, \leq)$ is called join-complete lattice if $\bigvee S$ exists for all subset S of L.

2 The Lattice of fractions

In this section, we will review the construction of a new algebraic structure studied by the authors. Let R be a non-empty distributive lattice and S be a non-empty subset of R, which is a complete meet-semilattice. We will see that the set of fractions $S^{-1}R$ is a lattice and it inherits many lattice properties from R. For more details see [7].

Definition 2.1. Define a binary relation $\sim_S$ on $R \times S$ by $(a, b) \sim_S (c, d)$ if and only if there exists $t \in S$ such that $(a \wedge d) \wedge t = (b \wedge c) \wedge t$.

Theorem 2.2. The relation $\sim_S$ on $R \times S$ is an equivalence relation.

Notation 2.3. The set of all equivalence classes of $\sim_S$ is denoted by $R/\sim_S$. In other words, $R/\sim_S = \{[(a,b)]_{\sim_S} : a \in R, b \in S\}$.

Lemma 2.4. (i) If $[(a,b)]_{\sim_S}$ and $[(a,c)]_{\sim_S}$ are two elements of $R/\sim_S$, then $[(a,b)]_{\sim_S} = [(a,c)]_{\sim_S}$.
(ii) Let $m = \bigwedge_{x\in S} x$. Then:
(1) $(a,m) \sim_S (b,m) \Longleftrightarrow (a,m) \sim_{\{m\}} (b,m)$,
(2) $R/\sim_S = R/\sim_{\{m\}}$.

Theorem 2.5. Let R be a non-empty distributive lattice, S_1 and S_2 be non-empty subsets of R which are complete meet-semilattices and $\bigwedge_{x\in S_1} x = \bigwedge_{x\in S_2} x$. Then, $R/\sim_{S_1} = R/\sim_{S_2}$.

Remark 2.6. (i) From now on, $R/\sim_S$ will be denoted by $S^{-1}R$ and it is called the fractions of lattice R with respect to S. Any element $[(a,b)]_{\sim_S} \in S^{-1}R$ is shown by a/b.
(ii) By Lemma 2.4.(ii), we can consider every S as a singleton $\{m\}$, where $m = \bigwedge_{x\in S} x$. Therefore, from now on we assume S to be the singleton $\{m\}$. So, by (i) we can write a/m for a/b.
(iii) For a/m and $b/m \in S^{-1}R$ we have $a/m = b/m$ if and only if $a \wedge m = b \wedge m$.

Lemma 2.7. $(S^{-1}R, \leq)$ is a partial ordered set, where $\leq$ is defined as follows:

$$a/m \leq b/m \Leftrightarrow a \wedge m \leq b \wedge m.$$

The well-defined binary operations $\vee, \wedge : S^{-1}R \times S^{-1}R \longrightarrow S^{-1}R$ are given by

$$a/m \vee b/m = (a \vee b)/m,$$

and

$$a/m \wedge b/m = (a \wedge b)/m.$$

Theorem 2.8. Let R be a join-complete lattice which satisfies JID and $S = \{m\}$. Then, $S^{-1}R$ is so.

Proof: One can easily verifies that $\bigvee(a_i/m) = (\bigvee a_i)/m$. Hence, $S^{-1}R$ is join-complete. Moreover, $a/m \wedge (\bigvee(b_i/m)) = a/m \wedge ((\bigvee b_i)/m) = (a \wedge \bigvee b_i)/m = (\bigvee(a \wedge b_i))/m = \bigvee((a \wedge b_i)/m) = \bigvee(a/m \wedge b_i/m)$. Therefore, $S^{-1}R$ satisfies JID. □

Theorem 2.9. Let $(R, \leq)$ be a distributive lattice, $S = \{m\}$ and I be an ideal of R. Then $S^{-1}I$ is an ideal of $S^{-1}R$. Moreover, any ideal of $S^{-1}R$ can be represented as $S^{-1}I$ where I is an ideal of R.

Proof: Straightforward. □

3 A representation of a class of Heyting algebras by fractions

In this section, we will see an application of fractions of a lattice, by giving a representation of a class of pseudo-Boolean lattices (or Heyting algebras) (Theorem 3.8). The following characterization for a finite Heyting algebra is well-known.

Theorem 3.1. ([4]) Let H be a finite lattice. Then, H is a Heyting algebra if and only if it is distributive.

If L is an infinite lattice we have Theorem 3.2, where a characterization of a pseudo-Boolean lattices is given.

Theorem 3.2. Let $(L, \leq)$ be a join-complete lattice. Then, L is pseudo-Boolean if and only if it is distributive and satisfies JID.

Proof: Let L be a pseudo-Boolean lattice. Then, it is distributive by Theorem 1.7(i). Clearly, $\bigvee_{i\in I}(a \wedge b_i) \leq a \wedge (\bigvee_{i\in I} b_i)$ for any arbitrary index set I. Now, let $\bigvee_{i\in I}(a \wedge b_i) = t$. We have :
$\forall i \in I,\ \ a \wedge b_i \leq t \Longrightarrow \forall i \in I,\ \ b_i \leq t : a \Longrightarrow \bigvee_{i\in I} b_i \leq t : a \Longrightarrow a \wedge (\bigvee_{i\in I} b_i) \leq t$. Hence, JID holds.
Conversely, let L be a distributive lattice which satisfies JID. Suppose, $a, b \in L$ are given. Since L is join-complete, $c = \bigvee_{a\wedge y\leq b} y$ exists and satisfies Definition 1.3. Therefore, L is pseudo-Boolean. □

The following example shows that JID in Theorem 3.2 is crucial.

Example 3.3. Let $(L, \subseteq)$ be the complete lattice of all closed subsets of the plane in Example 1.10. It is shown that L does not satisfy JID. Suppose, L is a pseudo-Boolean lattice. Let a, b be circles $x^2 + y^2 = 2$ and $x^2 + y^2 = 1$, respectively. Since L is pseudo-Boolean, for a, b there exists c such that it satisfies Definition 1.3. Let d be a closed subset such that $a \wedge d = \varphi$. Then, $x = b \vee d$ satisfies $a \wedge x \leq b$. Hence, $b \vee d \leq c$ for all d such that $a \wedge d = \varphi$. Therefore, $b \vee (\bigvee_{a\wedge d=\varphi} d) = \bigvee_{a\wedge d=\varphi}(b \vee d) \leq c$. Let $x = b \vee (\bigvee_{a\wedge d=\varphi} d)$. On the other hand, $\bigvee_{a\wedge d=\varphi} d = R^2 \backslash \{(x, y) | x^2 + y^2 < 2\}$ and clearly x does not satisfy $a \wedge x \leq b$.

Corollary 3.4. Let R be a join-complete ring of sets. Then, $S^{-1}R$ is pseudo-Boolean for all S, where $S = \{m\}$ and $m \in R$.

Proof: Clearly, any join-complete ring of sets satisfies JID. Hence, $S^{-1}R$ is pseudo-Boolean by Theorems 2.8 and 3.2. □

Definition 3.5. Let $(L, \leq)$ be a lattice. Then, L satisfies the join-completed prime ideals (JCPI) condition if every prime ideal of L is join-completed.

Example 3.6. (i) Every finite lattice satisfies the JCPI condition.

(ii) Let N be the chain of natural numbers with usual ordering $\leq$. Then, N satisfies the JCPI condition.
(iii) Let $I = [0,1]$ be the chain of real numbers between 0 and 1. Then, I does not satisfy the JCPI condition. Since $[0,a)$ is a prime ideal of I for all $a \in (0,1]$, and clearly it is not join-completed.

Theorem 3.7. A lattice L satisfies the JCPI condition if and only if every ideal of L is join-completed.

Proof: Let I be an ideal of L. Then, $I = \bigcap_{P\supseteq I} P$, by Corollary 1.12. Now, let $\{a_j\}_{j\in J}$ be an arbitrary family of elements of I. Then, $\{a_j\}_{j\in J} \subseteq \bigcap_{P\supseteq I} P$. Hence, $\{a_j\}_{j\in J} \subseteq P$ for all $P \supseteq I$. Since L satisfies the JCPI condition, $\bigvee_{j\in J} a_j \in P$ for all $P \supseteq I$. Therefore, $\bigvee_{j\in J} a_j \in I = \bigcap_{P\supseteq I} P$ and consequently, I is join-completed. Clearly, if every ideal of L is join-completed, then L satisfies the JCPI condition. □

Now, we can give a new representation of a pseudo-Boolean lattice which satisfies the JCPI condition, based on our recent results by lattice of fractions [7].

Theorem 3.8. Let $(L,\leq)$ be a pseudo-Boolean lattice which satisfies the JCPI condition. Then, it is isomorphic to a lattice of fractions $S^{-1}R$, where R is a join-complete ring of sets.

Proof: Let L be a pseudo-Boolean lattice which also satisfies the JCPI condition. Now, let $X = \{P : P \text{ be a prime ideal of L}\}$. For $a \in L$, define $r(a) = \{P \in X : a \notin P\}$ and let $R = \{r(a) : a \in L\}$. We claim that the ring of sets $(R,\subseteq)$, is join-completed. To do this end we show that $\bigcup_{i\in I} r(a_i) = r(\bigvee_{i\in I} a_i)$. Clearly, $\bigcup_{i\in I} r(a_i) \subseteq r(\bigvee_{i\in I} a_i)$. Now, let $P \in r(\bigvee_{i\in I} a_i)$. Hence, $\bigvee_{i\in I} a_i \notin P$. Since L satisfies the JCPI condition, there exists $i_0 \in I$ such that $a_{i_0} \notin P$. Hence, $P \in r(a_{i_0}) \subseteq \bigcup_{i\in I} r(a_i)$. Therefore, R is join-completed. Now, L is isomorphic to the ring of sets R as in the proof of Theorem 1.13 in [4]. As every pseudo-Boolean lattice is bounded with the upper bound $1 \in L$ we have $X = r(1) \in R$. Define $\phi : L \longrightarrow S^{-1}R$ by $a \longmapsto r(a)/X$. It can easily verified that ϕ is a lattice isomorphism. □

Remark 3.9. Note that if R is a join-complete ring of sets, then $S^{-1}R$ is pseudo-Boolean, by Theorem 2.8. On the other hand, R may not satisfy the JCPI. For example, let R be the complete lattice (and of course, ring of sets) of all subsets of $[0,1]$ and let P be $\mathcal{P}\ ([0,1/2))\backslash\{[0,1/2)\}$. Clearly, P is a prime ideal of R which is not join-complete.

Remark 3.10. By example given in Remark 3.9, the converse of Theorem 3.8 may not hold, generally. However, in finite case, every join-semilattice satisfies the JCPI and we have the following theorem, which is extended version of Theorem 5.4 of [7].

Theorem 3.11. Let L be a finite lattice. Then, L is pseudo-Boolean lattice if and only if it is isomorphic to a lattice of fractions $S^{-1}R$, where R is a finite ring of sets.

We close this section by the following question.

Open problem. It is still interesting to find a characterization for Heyting algebras by means of fractions, in general, without JCIP condition.

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