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A New Hilbert Type Inequality with the Best Constant

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Abstract. In this paper it is shown that a new Hilbert type inequality can be established by introducing a proper logarithm function. And the constant factor is proved to be the best possible . As applications, some inequalities which they are equivalent each other are built.

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1. 1ntroduction

Let $f(x), g(x) \in L^2(0, +\infty)$. Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \le \pi \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}. \quad (1.1)$$

where the coefficient π is the best possible. And the equality in (1.1) hold if and only if $f(x)=0$, or $g(x)=0$. This is the famous Hilbert integral inequality (see [3,7]).

Owing to the importance of the Hilbert inequality and the Hilbert type inequality in analysis and applications, some mathematicians have been studying them. Recently, various improvements and extensions of (1.1) appear in a great deal of papers (see [1, 2, 4, 5, 9, 10] etc.). The aim of the present paper is to build a Hilbert type integral inequality by introducing a proper integral kernel function and by using the technique of analysis, and to discuss the constant factor of which is related to the Euler number, and then to study some equivalent forms of them.

In the sake of convenience, we introduce some notations and define some functions.

Let $0<\alpha<1$ and n be a positive integer. Define a function ζ^* by

$$\zeta^*(n,\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(\alpha+k)^n}. \tag{1.2}$$

And further define the function ζ_2 by

$$\zeta_2=(2n)!\left\{2\zeta^*(2n+1,\tfrac{1}{2})\right\}, \quad (n\in N_0) \tag{1.3}$$

In order to prove our main results, we need the following lemmas.

Lemma 1.1. Let $0<\alpha<1$ and n be a nonnegative integer. Then

$$\int_0^1 t^{\alpha-1}\left(\ln\tfrac{1}{t}\right)^n\tfrac{1}{1+t}\,dt=n!\zeta^*(n+1,\alpha). \tag{1.4}$$

where ζ^* is defined by (1.2).

This result has been given in the paper [6]. Hence its proof is omitted here.

Lemma 1.2. With the assumptions as Lemma 1.1, then

$$\int_0^\infty u^{\alpha-1}\left(\ln\tfrac{1}{u}\right)^{2n}\tfrac{1}{1+u}\,du=(2n)!\left\{\zeta^*\left(2n+1,\alpha\right)+\zeta^*\left(2n+1,1-\alpha\right)\right\}, \tag{1.5}$$

where ζ^* is defined by (1.2).

Proof. It is easy to deduce that

$$\int_0^\infty u^{\alpha-1}\left(\ln\tfrac{1}{u}\right)^{2n}\tfrac{1}{1+u}\,du=\int_0^1 u^{\alpha-1}\left(\ln\tfrac{1}{u}\right)^{2n}\tfrac{1}{1+u}\,du+\int_1^\infty u^{\alpha-1}\left(\ln\tfrac{1}{u}\right)^{2n}\tfrac{1}{1+u}\,du$$

$$=\int_0^1 u^{\alpha-1}\left(\ln\tfrac{1}{u}\right)^{2n}\tfrac{1}{1+u}\,du+\int_0^1 v^{-\alpha}\left(\ln v\right)^{2n}\tfrac{1}{1+v}\,dv$$

$$=\int_0^1 u^{\alpha-1}\left(\ln\tfrac{1}{u}\right)^{2n}\tfrac{1}{1+u}\,du+\int_0^1 v^{(1-\alpha)-1}\left(\ln\tfrac{1}{v}\right)^{2n}\tfrac{1}{1+v}\,dv.$$

By using Lemma 1.1, the equality (1.5) is obtained at once.

Throughout the paper, we define $\left(\ln \frac{x}{y}\right)^0 = 1$, when $x = y$.

2. Main Results

We are ready now to formulate our main results.

Theorem 2.1. Let f and g be two real functions, **a**nd n be a nonnegative integer, If

$\int_0^\infty f^2(x)dx < +\infty$ and $\int_0^\infty g^2(x)dx < +\infty$, then

$$\int_0^\infty\int_0^\infty \frac{(\ln\frac{x}{y})^{2n} f(x)g(y)}{x+y}dxdy \le \left(\pi^{2n+1}E_n\right)\left\{\int_0^\infty f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)dx\right\}^{\frac{1}{2}}, \qquad (2.1)$$

where the constant factor $\pi^{2n+1}E_n$ is the best possible, and that $E_0 = 1$ and $E_{n's}$ are the Euler numbers , viz. $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, $E_4 = 1385$, $E_5 = 50521$, etc**.** And the equality holds if and only if $f(x) = 0$, or $g(x) = 0$.

Proof. We may apply the Cauchy inequality to estimate the left-hand side of (2.1) as follows**:**

$$\int_0^\infty\int_0^\infty \frac{(\ln\frac{x}{y})^{2n} f(x)g(y)}{x+y}dxdy = \int_0^\infty\int_0^\infty\left(\frac{(\ln\frac{x}{y})^{2n}}{x+y}\right)^{\frac{1}{2}}\left(\frac{x}{y}\right)^{\frac{1}{4}} f(x)\left(\frac{(\ln\frac{x}{y})^{2n}}{x+y}\right)^{\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{1}{4}} g(y)dxdy$$

$$\le\left\{\int_0^\infty\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n}}{x+y}\left(\frac{x}{y}\right)^{\frac{1}{2}} f^2(x)dxdy\right\}^{\frac{1}{2}}\left\{\int_0^\infty\int_0^\infty \frac{(\ln\frac{x}{y})^{2n}}{x+y}\left(\frac{y}{x}\right)^{\frac{1}{2}} g^2(y)dxdy\right\}^{\frac{1}{2}}$$

$$=\left(\int_0^\infty \omega(x) f^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty \omega(x)g^2(x)dx\right)^{\frac{1}{2}} \qquad (2.2),$$

where $\omega(x) = \int_0^\infty \frac{(\ln\frac{x}{y})^{2n}}{x+y}\left(\frac{x}{y}\right)^{\frac{1}{2}} dy$,

By using Lemma 1.2, it is easy to deduce that

$$\omega(x) = \int_0^\infty \frac{(\ln\frac{x}{y})^{2n}}{x\left(1+\frac{y}{x}\right)}\left(\frac{x}{y}\right)^{\frac{1}{2}} dy = \int_0^\infty u^{-\frac{1}{2}}(\ln\tfrac{1}{u})^{2n}\frac{1}{1+u}du = \zeta_2\,\mathbf{.} \qquad (2.3)$$

where ζ_2 is defined by (1.3). Based on (1.2) and (1.3), we have

$$\zeta_2 = (2n)!\left\{2\zeta^*(2n+1,\tfrac{1}{2})\right\} = (2n)!2\sum_{k=0}^{\infty}\frac{(-1)^k}{(\frac{1}{2}+k)^{2n+1}}$$

$$= (2n)!2^{2n+2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}}.$$

It is known from the paper [8] that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2}(2n)!}E_n. \tag{2.4}$$

where $E_{n's}$ are the Euler numbers ,

viz. $E_1 = 1,\ E_2 = 5,\ E_3 = 61,\ E_4 = 1385,\ E_5 = 50521,$ etc.

Since $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = \frac{\pi}{4}$, we can define $E_0 = 1$. As thus, the relation (2.4) is also valid when $n = 0$. So, we get from (2.3) and (2.4) that

$$\omega(x) = \pi^{2n+1}E_n \quad ,$$

(2.5)

It follows from (2.2) and (2.5) that the inequality (2.1) is valid.

If $f(x) = 0$ or $g(x) = 0$, the equality in (2.1) obviously holds . Consider the case $f(x)g(x) \neq 0$,

then we have $0 < \int_0^{\infty} f^2(x)dx < +\infty$ and $0 < \int_0^{\infty} g^2(x)dx < +\infty$. If (2.2) takes the form of the equality, then there exist a pair of non-zero constants c_1 and c_2 such that

$$c_1\frac{\left(\ln\frac{x}{y}\right)^{2n}}{x+y}\left(\frac{x}{y}\right)^{\frac{1}{2}} f^2(x) = c_2\frac{\left(\ln\frac{x}{y}\right)^{2n}}{x+y}\left(\frac{y}{x}\right)^{\frac{1}{2}} g^2(y) \qquad \text{a.e. on } (0,+\infty)\times(0,+\infty)$$

Then we have

$$c_1x\ f^2(x) = c_2y\ g^2(y) = C_0. \quad \text{(constant)} \qquad \text{a.e. on } (0,+\infty)\times(0,+\infty)$$

Without losing the generality, we suppose that $c_1 \neq 0$, then

$$\int_0^\infty f^2(x)\,dx = \frac{C_0}{C_1}\int_0^\infty x^{-1}dx.$$

This contradicts that $0 < \int_0^\infty f^2(x)\,dx < +\infty$. Hence it is impossible to take the equality in (2.2). It shows that it is also impossible to take the equality in (2.1) .

It remains to need only to show that $\pi^{2n+1}E_n$ in (2.1) is the best possible. $\forall\ 0<\varepsilon<1$.

Define two functions by

$$\tilde f(x) = \begin{cases} 0 & if \quad x\in(0,\ 1) \\ x^{-\frac{1+\varepsilon}{2}} & if \quad x\in[1,\infty) \end{cases} \text{ and } \tilde g(y) = \begin{cases} 0 & if \quad y\in(0,\ 1) \\ y^{-\frac{1+\varepsilon}{2}} & if \quad y\in[1,\infty) \end{cases}.$$

It is easy to deduce that

$$\int_0^{+\infty}\tilde f^2(x)\,dx = \int_0^{+\infty}\tilde g^2(y)\,dy = \frac{1}{\varepsilon}.$$

If $\pi^{2n+1}E_n$ is not the best possible, then there exists $C>0$, such that $C<\pi^{2n+1}E_n$ and

$$S(\tilde f,\tilde g) = \int_0^\infty\int_0^\infty \frac{(\ln\frac{x}{y})^{2n}\tilde f(x)\tilde g(y)}{x+y}dxdy \le C\left(\int_0^\infty \tilde f^2(x)\,dx\right)^{\frac12}\left(\int_0^\infty \tilde g^2(y)\,dy\right)^{\frac12} = \frac{C}{\varepsilon}.$$

(2.6)

On the other hand, we have

$$S(\tilde f,\tilde g) = \int_0^\infty\int_0^\infty \frac{\left\{x^{-\frac{1+\varepsilon}{2}}\right\}\left\{(\ln\frac{x}{y})^{2n}y^{-\frac{1+\varepsilon}{2}}\right\}}{x+y}dxdy = \int_1^\infty\left\{\int_1^\infty \frac{(\ln\frac{x}{y})^{2n}y^{-\frac{1+\varepsilon}{2}}}{x+y}dy\right\}\left\{x^{-\frac{1+\varepsilon}{2}}\right\}dx$$

$$= \int_1^\infty\left\{\int_{1/x}^\infty \frac{(\ln\frac{1}{u})^{2n}u^{-\frac{1+\varepsilon}{2}}}{1+u}du\right\}\left\{x^{-1-\varepsilon}\right\}dx$$

$$= \int_1^\infty\left\{\int_{1/x}^1 \frac{(\ln\frac{1}{u})^{2n}u^{-\frac{1+\varepsilon}{2}}}{1+u}du\right\}\left\{x^{-1-\varepsilon}\right\}dx + \int_1^\infty\left\{\int_1^\infty \frac{(\ln\frac{1}{u})^{2n}u^{-\frac{1+\varepsilon}{2}}}{1+u}du\right\}\left\{x^{-1-\varepsilon}\right\}dx$$

$$= \frac{1}{\varepsilon}\int_0^1 \frac{(\ln\frac{1}{u})^{2n}u^{-\frac{1-\varepsilon}{2}}}{1+u}du + \frac{1}{\varepsilon}\int_1^\infty \frac{(\ln\frac{1}{u})^{2n}u^{-\frac{1+\varepsilon}{2}}}{1+u}du \ . \tag{2.7}$$

When ε is small enough, we can write (2.7) in the following form:

$$S(\tilde{f},\tilde{g})=\frac{1}{\varepsilon}\left(\int_0^1\frac{(\ln\frac{1}{u})^{2n}u^{-\frac{1}{2}}}{1+u}du+\circ_1(1)\right)+\frac{1}{\varepsilon}\left(\int_1^\infty\frac{(\ln\frac{1}{u})^{2n}u^{-\frac{1}{2}}}{1+u}du+\circ_2(1)\right).$$

$$=\frac{1}{\varepsilon}\left(\int_0^\infty u^{-\frac{1}{2}}\left(\ln\frac{1}{u}\right)^{2n}\frac{1}{1+u}du+\circ(1)\right). \quad (\varepsilon\to 0) \qquad (2.8)$$

Based on (2.3), (2.5) and (2.8), we obtain

$$S(\tilde{f},\tilde{g})=\frac{1}{\varepsilon}\left\{(\pi^{2n+1}E_n)+\circ(1)\right\}, \qquad (\varepsilon\to 0) \qquad (2.9)$$

When ε is small enough, it is obvious that the inequality (2.6) is in contradiction with (2.9). Therefore, the constant factor $\pi^{2n+1}E_n$ in (2.1) is the best possible. Thus the proof of Theorem is completed.

Notice that the constant factor $\pi^{2n+1}E_n$ in (2.1) can be reduced to π^3, if $n=1$. Hence we have the following result.

Corollary 2.1. With the assumptions as Theorem 2.1, then

$$\int_0^\infty\int_0^\infty\frac{(\ln\frac{x}{y})^2 f(x)g(y)}{x+y}dxdy \le \pi^3\left\{\int_0^\infty f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)dx\right\}^{\frac{1}{2}}. \quad (2.10)$$

where the constant factor π^3 is the best possible. And the equality holds in (2.10) if and only if $f(x)=0$, or $g(x)=0$.

3 . Some Applications

As applications, we will build the following inequalities.

Theorem 3.1. Let n be a nonnegative integer. If $\int_0^\infty f^2(x)dx<+\infty$, then

$$\int_0^\infty\left\{\int_0^\infty\frac{(\ln\frac{x}{y})^{2n}}{x+y}f(x)dx\right\}^2 dy\le\left(\pi^{2n+1}E_n\right)^2\int_0^\infty f^2(x)dx \quad ,$$

(3.1)

where $\left(\pi^{2n+1}E_n\right)^2$ in (3.1) is the best possible, and that $E_0=1$ and E_n is the Euler number , viz. $E_1=1$,

$E_2 = 5,\ E_3 = 61,\ E_4 = 1385, E_5 = 50521,$ etc. And the inequality (3.1) is equivalent to (2.1). And the equality holds if and only if $f(x) = 0$.

Proof. Assume that the inequality (2.1) is valid. Setting a real function $g(y)$ as

$$g(y) = \int_0^\infty \frac{(\ln\frac{x}{y})^{2n}}{x+y} f(x)\,dx, \qquad y \in (0, +\infty)$$

By using (2.1), we have

$$\int_0^\infty \left\{ \int_0^\infty \frac{(\ln\frac{x}{y})^{2n}}{x+y} f(x)\,dx \right\}^2 dy = \int_0^\infty \int_0^\infty \frac{(\ln\frac{x}{y})^{2n}}{x+y} f(x) g(y)\,dxdy$$

$$\le \left(\pi^{2n+1} E_n\right) \left\{ \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y)\,dy \right\}^{\frac{1}{2}}$$

$$= \left(\pi^{2n+1} E_n\right) \left\{ \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty \left(\int_0^\infty \frac{(\ln\frac{x}{y})^{2n}}{x+y} f(x)\,dx \right)^2 dy \right\}^{\frac{1}{2}} \qquad (3.2)$$

where $E_0 = 1$ and E_n is the Euler number , viz. $E_1 = 1$, $E_2 = 5,\ E_3 = 61,\ E_4 = 1385, E_5 = 50521$, etc.

It follows from (3.2) that the inequality (3.1) is valid after some simplifications.

On the other hand, assume that the inequality (3.1) keeps valid, by applying in turn Cauchy's inequality and (3.1), we have

$$\int_0^\infty \int_0^\infty \frac{(\ln\frac{x}{y})^{2n}}{x+y} f(x) g(y)\,dxdy = \int_0^\infty \left\{ \int_0^\infty \frac{(\ln\frac{x}{y})^{2n}}{x+y} f(x)\,dx \right\} g(y)\,dy$$

$$\le \left\{ \int_0^\infty \left(\int_0^\infty \frac{(\ln\frac{x}{y})^{2n}}{x+y} f(x)\,dx \right)^2 dy \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y)\,dy \right\}^{\frac{1}{2}}$$

$$\le \left\{ \left(\pi^{2n+1} E_n\right)^2 \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y)\,dy \right\}^{\frac{1}{2}}$$

$$= \left(\pi^{2n+1} E_n\right) \left\{ \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y)\,dy \right\}^{\frac{1}{2}} \qquad (3.3)$$

where $E_0 = 1$ and E_n is the Euler number , viz. $E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, E_5 = 50521$, etc.

If the constant factor $\left(\pi^{2n+1}E_n\right)^2$ in (3.1) is not the best possible, then it is known from (3.3) that the constant factor $\pi^{2n+1}E_n$ in (2.1) is also not the best possible. This is a contradiction . It is obvious that the equality holds in (3.1) if and only if $f(x) = 0$. Theorem is proved.

Corollary 3.1. With the assumptions as Theorem 3.1, then

$$\int_0^\infty \left\{ \int_0^\infty \frac{(\ln\frac{x}{y})^2}{x+y} f(x)\,dx \right\}^2 dy \le \pi^6 \int_0^\infty f^2(x)\,dx, \tag{3.4}$$

where the constant factor π^6 is the best possible. Inequality (3.4) is equivalent to (2.10). A n And the equality holds in (3.1) if and only if $f(x) = 0$.

The proof of Corollary 3.2 is similar to one of Theorem 3.1, it is omitted here.

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