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Multiplier on Character Amenable Banach Algebras

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Abstract

In this paper we prove that for a commutative character amenable Banach algebra $\mathcal{A}$, if $T : \mathcal{A} \to \mathcal{A}$ is a multiplier then T has closed range if and only if $T = BP = PB$, where $B \in M(A)$ is invertible and $p \in M(A)$ is idempotent. By this result we characterize each multiplier with closed range on such Banach algebra (proposition 3.7), and so we get a necessary condition for character amenability of algebras.

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1 Introduction

Let $\mathcal{A}$ be a commutative Banach algebra. A map $T : \mathcal{A} \to \mathcal{A}$ is said to be a multiplier if it satisfies $xT(y) = T(x)y$ for all $x, y \in \mathcal{A}$. We denote the set of all multiplier on $\mathcal{A}$ by $M(\mathcal{A})$. If $\mathcal{A}$ has a bounded approximate identity, then $M(\mathcal{A})$ is a closed subalgebra of $B(\mathcal{A})$ (The Banach algebra of all bounded operators on $\mathcal{A}$), and in this case for $T \in M(\mathcal{A})$ we have

$$T(xy) = xT(y) = T(x)y \qquad for\ all\ x,\ y \in \mathcal{A}.$$

When the range of a multiplier T on a commutative Banach algebra is closed, is one of the important question in the theory of Banach algebras. Obviously, if T is a product of an idempotent multiplier P and an invertible multiplier B, then $T(\mathcal{A})$ is closed. So the question modifies to the latter assertion, and if we find the equivalent condition for the factorization of T as above, then the question to be solvable.

For example, as we know, for a large class of Banach algebras $\mathcal{A}$, the statement; $T(\mathcal{A})$ has a bounded approximate identity or $\mathcal{A} = T(\mathcal{A}) \oplus Ker(T)$, are equivalent to factorization of T. (For more details see [1], [9], [13] and [14]).

The starting point of this subject is done by Host and Parreau in 1971. They showed that if G is locally compact group, then for $\mu \in M(G)$ the ideal $\mu * L^1(G)$ in group algebra $L^1(G)$ is closed if and only if μ is a product of an idempotent and invertible measure.

In this paper, we present the concept of character amenable Banach algebra, which was introduced by Lau, Pym and Kaniuth, and we investigate the multipliers on it.

In Theorem 3.4 we show that for a commutative character amenable Banach algebra $\mathcal{A}$, a multiplier T on $\mathcal{A}$ has closed range if and only if T factors as a product of an invertible and idempotent multipliers.

2 Notation and preliminaries

In this section we have collected some notation and results which are needed for the subsequent sections. Given a commutative complex Banach algebra $\mathcal{A}$ with or without identity, let $\Delta(\mathcal{A})$ stand for the *spectrum* of $\mathcal{A}$, i.e. the set of all nontrivial multiplicative linear functionals on $\mathcal{A}$. For each $a \in \mathcal{A}$, let $\widehat{a} : \Delta(\mathcal{A}) \to C$ denotes the corresponding *Gelfand transform* given by $\widehat{a}(\varphi) := \varphi(a)$ for all $\varphi \in \Delta(\mathcal{A})$. On $\Delta(\mathcal{A})$ we shall have to consider both the *Gelfand* and the *hull-kernel topology*. The latter is determined by the Kuratowski closure operation $cl(E) := hull(ker(E)) := \{\psi \in \Delta(\mathcal{A}) \ : \ \psi(u) = 0 \ for\ all\ u \in \mathcal{A}\ with\ \varphi(u) = 0\ for\ each\ \varphi \in E\}$ for all $E \subseteq \Delta(\mathcal{A})$. The hull-kernel topology is always coarser than the Gelfand topology on $\Delta(\mathcal{A})$, and they coincide if and only if the algebra $\mathcal{A}$ is regular. For further information concerning to the hull-kernel topology, we refer to [11].

Now let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. We denote the first and second dual of $\mathcal{A}$ by $\mathcal{A}^*$ and $\mathcal{A}^{**}$, respectively. The *first Arens multiplication* on $\mathcal{A}^{**}$ is defined by three steps as follows. For a, b in $\mathcal{A}$, f in $\mathcal{A}^*$ and m, n in $\mathcal{A}^{**}$, the elements $f.a$, $n.f$ of $\mathcal{A}^*$ and $m.n$ of $\mathcal{A}^{**}$ are defined by

$$\langle f.a, b\rangle = \langle f, a.b\rangle, \ \langle n.f, a\rangle = \langle n, f.a\rangle, \ \langle m.n, f\rangle = \langle m, n.f\rangle$$

This multiplications have the basic properties as follows: For a fixed n in $\mathcal{A}^{**}$, the mapping $m \mapsto m.n$ is weak*-weak*-continuous. But, the mapping $n \mapsto m.n$ may not be weak*-weak*- continuous, unless m be in $\mathcal{A}$.

Also, by an easy application of the Goldstine's theorem, we have a double-limit definition

$$m.n = w^* - lim_\alpha w^* - lim_\beta \; a_\alpha b_\beta$$

where $(a_\alpha)_{\alpha\in I}$ and $(b_\beta)_{\beta\in J}$ are two bounded nets in $\mathcal{A}$ converging in the weak*-topology of $\mathcal{A}^{**}$ to m and n, respectively. If now $T : \mathcal{A} \to \mathcal{A}$ is a multiplier on A then, as follows readily from the double-limit definition of the product $m.n$, for all m and n in $\mathcal{A}^{**}$, we have

$$m.T^{**}(n) = T^{**}(m.n) = T^{**}(m).n,$$

so T^{**} is also a multiplier on $\mathcal{A}^{**}$. If $(e_\alpha)_{\alpha\in I}$ is a bounded approximate identity in $\mathcal{A}$ then each weak*-cluster point e in $\mathcal{A}^{**}$ of this net, is a right identity in $\mathcal{A}$ so that, for each $m \in \mathcal{A}^{**}$, $m.e = m$. In particular, for each $a \in \mathcal{A}$, $a.e = a$, and the important equality, $T^{**}(m) = m.T^{**}(e)$ holds for all $m \in \mathcal{A}^{**}$.

In [7], Lau, Pym and Kaniuth introduced and investigated a large class of Banach algebras which they called *φ-amenable* Banach algebras. Given $\varphi \in \Delta(\mathcal{A})$, a Banach algebra $\mathcal{A}$ is said to be φ-amenable if there exists $m \in \mathcal{A}^{**}$ such that $\langle m, \varphi\rangle = 1$ and $\langle m, f.a\rangle = \varphi(a)\langle m, f\rangle$ for all $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$. A commutative Banach algebra $\mathcal{A}$ is said to be *character amenable*, if $\mathcal{A}$ has a bounded approximate identity and for each $\varphi \in \Delta(\mathcal{A}) \cup \{0\}$, $\mathcal{A}$ is φ-amenable. Here we mention some of the well-known properties of these algebras that we shall need. By theorem 1.4. of [7], for $\varphi \in \Delta(\mathcal{A})$ the Banach algebra $\mathcal{A}$ is φ-amenable if and only if there exists a bounded net $(u_\alpha)_\alpha$ in $\mathcal{A}$ such that $\|au_\alpha - \varphi(a)u_\alpha\| \to 0$ for all $a \in \mathcal{A}$ and $\varphi(u_\alpha) = 1$ for all α. Also, a commutative Banach algebra is character amenable if and only if for each $\varphi \in \Delta(\mathcal{A})$, the ideal $\ker\varphi$ has a bounded approximate identity, see [7, corollary 2.3].

3 Closed range multipliers on Character amenable Banach algebra

Our aim of this section is to prove that a multiplier T on a commutative character amenable Banach algebra has a closed range if and only if T is the product of an idempotent multiplier P and an invertible multiplier B (i.e. $T = P \circ B$).

We start this section by the following lemma, which plays a crucial role in the study of the structure of multipliers on a commutative character amenable Banach algebra with closed range.

Lemma 3.1 *If $\mathcal{A}$ is a character amenable Banach algebra, and $T : \mathcal{A} \to \mathcal{A}$ is a multiplier with closed range, then for each $\varphi \in \Delta(T(\mathcal{A}))$ the Banach algebra $T(\mathcal{A})$ is φ-amenable.*

Proof. For arbitrary $\varphi \in \Delta(T(\mathcal{A}))$ we can choose $b \in \mathcal{A}$ for which $\varphi(T(b)) = 1$. If now define the linear functional $\widetilde{\varphi}$ on $\mathcal{A}$ by $\widetilde{\varphi}(a) := \varphi(T(b)a)$ for $a \in \mathcal{A}$, then $\widetilde{\varphi}$ is multiplicative and non-zero, and the definition of $\widetilde{\varphi}$ is independent of the choice of b. Therefore $\widetilde{\varphi} \in \Delta(\mathcal{A})$. As we mentioned in preliminaries, by $\widetilde{\varphi}$-amenability of $\mathcal{A}$, there exist a net $(u_\alpha)_{\alpha\in I}$ in $\mathcal{A}$ such that $\widetilde{\varphi}(u_\alpha) = 1$ for all $\alpha \in I$, and $\|au_\alpha - \widetilde{\varphi}(a)u_\alpha\| \to 0$ for each $a \in \mathcal{A}$.
Now for each $\alpha \in I$, set $\nu_\alpha := T(b)u_\alpha$. So we have $\varphi(\nu_\alpha) = 1$ and for each $a \in \mathcal{A}$

$$\|T(a)\nu_\alpha - \varphi(T(a))\nu_\alpha\| \leq \|T(b)\|.\|T(a)u_\alpha - \widetilde{\varphi}(T(b))u_\alpha\| \to 0$$

and this complete the proof.

Now we are going to show that for a closed range multiplier T on a commutative character amenable Banach algebra $\mathcal{A}$, the Banach algebra $T(A)$ has a bounded approximate identity. For this end we need state some definitions. Given a Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule X, a continuous derivation of $\mathcal{A}$ to X, or X-derivation is a continuous linear mapping D from $\mathcal{A}$ into X such that $D(ab) = D(a).b + a.D(b)$ for all $a, b \in \mathcal{A}$. For each $x \in X$, the mapping $D_x : \mathcal{A} \to X$ defined by $D_x(a) = a.x - x.a$ is a bounded X-derivation, called the inner derivation associated with x. We denote the space of all continuous X-derivations by $Z^1(\mathcal{A}, X)$ and the subspace of all inner derivations in X by $N^1(\mathcal{A}, X)$. The quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$ is called the first continuous cohomology group of $\mathcal{A}$ with coefficients in X. Therefore if $H^1(\mathcal{A}, X) = \{0\}$, then every continuous X-derivation is inner.

The following theorem is known; $(a) \Leftrightarrow (b)$ and $(b) \Leftrightarrow (c)$ were proved in [7] and [5] respectively.

Theorem 3.2 *Let $\mathcal{A}$ be a commutative Banach algebra and $T : \mathcal{A} \to \mathcal{A}$ be a multiplier with closed range. Then the following assertion are equivalent.*

(a) For each $\varphi \in \Delta(T(\mathcal{A})) \cup \{0\}$ the Banach algebra $T(\mathcal{A})$ is φ-amenable.

(b) For each $\varphi \in \Delta(T(\mathcal{A})) \cup \{0\}$, if X is a Banach $T(\mathcal{A})$-bimodule such that $T(a).x = \varphi(T(a)).x$ for all $x \in X$ and $a \in \mathcal{A}$, then $H^1(T(\mathcal{A}), X^*) = \{0\}$;

(c) $T(\mathcal{A})$ has a bounded approximate identity.

By combination of Lemma 3.1 and Theorem 3.2 we have the following result, that is an important consequence of Lemma 3.1.

Theorem 3.3 *Let $\mathcal{A}$ be a commutative character amenable Banach algebra and $T : \mathcal{A} \to \mathcal{A}$ be a multiplier with closed range. Then the Banach algebra $T(\mathcal{A})$ has a bounded approximate identity.*

Now we are ready to state and prove the following theorem as the main result of this section.

Theorem 3.4 *Let $T : \mathcal{A} \to \mathcal{A}$ be a multiplier on a commutative character amenable Banach algebra $\mathcal{A}$. Then the following statements are equivalent:*
(a) T has closed range.
(b) $T(\mathcal{A})$ has a bounded approximate identity.
(c) $T^2(\mathcal{A}) = T(\mathcal{A})$
(d) $\mathcal{A} = T(\mathcal{A}) \oplus Ker(T)$
(e) $T = BP = PB$, where $B \in M(A)$ is invertible and $p \in M(A)$ is idempotent.

Proof. (a) implies (b) by theorem 3.3. Suppose that (b) holds. Then by Cohen's factorization theorem we have $T^2(\mathcal{A}) = T^2(\mathcal{A}\mathcal{A}) = T(\mathcal{A})T(\mathcal{A}) = T(\mathcal{A})$. So (b)$\Rightarrow$(c).
Now it is easy to see that the hypothesis (c) implies that $\mathcal{A} = T(\mathcal{A}) + Ker(T)$. Therefore the implication $(c) \Rightarrow (d)$ follows if we show that $T(\mathcal{A}) \cap Ker(T) = \{0\}$. For this end, if $T(z) = x \in Ker(T)$, then $xT(\mathcal{A}) = xKer(T) = \{0\}$. Thus, since $\mathcal{A}$ has a bounded approximate identity we have $x = 0$.
Now, suppose that $\mathcal{A} = T(\mathcal{A}) \oplus Ker(T)$. Since in this case $T^2(\mathcal{A}) = T(\mathcal{A})$ and $Ker(T) = Ker(T^2)$, T is a bijection on $T(\mathcal{A})$. Therefore, the linear operator B on $\mathcal{A}$, defined by $B(a + b) := T(a) + b$ for all $a \in T(\mathcal{A})$ and $b \in Ker(T)$, is obviously bijective. Moreover, let P be the linear projection on $\mathcal{A}$ defined by $P(a + b) = a$ for all $a \in T(\mathcal{A})$ and $b \in Ker(T)$, it is straightforward to see that $T = PB = BP$. Thus (d) implies (e). That (e) implies (a) is trivial.

Remark 3.5 It should be noted that there are many Banach algebras which satisfy the hypothesis of theorem 3.4. For example it contains all the C^*-algebras and all the commutative semisimple amenable Banach algebras. Also, there is no well-known Banach algebra which is not character amenable, but each multiplier T on it satisfies the five equivalent statements of theorem 3.4.

Theorem 3.4 gives us a necessary condition for character amenability of Banach algebras.

Example: Let $\mathcal{A}$ be the classical disk algebra and $T : \mathcal{A} \to \mathcal{A}$ be the multiplier defined by $T(f)(z) = z.f(z)$. Then $T(\mathcal{A}) = \{f \in \mathcal{A} : \ f(0) = 0\}$ and $T^2(\mathcal{A}) = \{f \in \mathcal{A} : \ f(0) = f'(0) = 0\}$. The ideal $T(\mathcal{A})$ is closed in $\mathcal{A}$, but $T^2(\mathcal{A})$ is not dense in $T(\mathcal{A})$, and the closed ideal $T(\mathcal{A})$ does not have a bounded approximate identity. Therefore, by 3.4, the classical disk algebra is not character amenable.

Laursen and Mbekhta in [9] studied on the properties of closed range multiplier on a semisimple Banach algebra and they showed that a multiplier on such an algebras has a closed range if and only if $dist(0, \sigma(T) \setminus \{0\}) > 0$. Therefore, as a corollary of the Theorem 3.4, we can prove a special spectral properties of a multiplier with closed range on commutative character amenable Banach algebras.

Corollary 3.6 *Let $\mathcal{A}$ be a semisimple commutative character amenable Banach algebra. Then a multiplier $T : \mathcal{A} \to \mathcal{A}$ has a closed range if and only if zero is an isolated point of its spectrum $\sigma(T)$.*

By Theorem 3.4(d) for a commutative character amenable Banach algebra $\mathcal{A}$, a multiplier $T : \mathcal{A} \to \mathcal{A}$ with closed range is one-to-one if and only if it is surjective. Therefore we can characterize each multiplier with closed range on such Banach algebra as follows.

Proposition 3.7 *Let $\mathcal{A}$ be a commutative semisimple character amenable Banach algebra. If the Gelfand spectrum of $\mathcal{A}$ is connected. Then a nonzero multiplier $T : \mathcal{A} \to \mathcal{A}$ has a closed range if and only if T is invertible.*

Proof. By theorem 3.4(d), we have a decomposition for $\Delta(\mathcal{A})$ into the disjoint hulls $h(T(\mathcal{A}))$ and $h(\ker T)$. It follows from the connectedness of $\Delta(\mathcal{A})$ that either $h(T(\mathcal{A})) = \Delta(\mathcal{A})$ or $h(\ker T) = \Delta(\mathcal{A})$. Therefore, since T is non-trivial, and $\mathcal{A}$ is semisimple, we conclude that T is invertible.

The following corollary is a special case of proposition 3.7.

Corollary 3.8 *Let $\mathcal{A}$ be a commutative semisimple character amenable Banach algebra. If the Gelfand spectrum of $\mathcal{A}$ is connected, then $\mathcal{A}$ has no proper closed principal ideal.*

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