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Rings over which all Gorenstein Flat Modules are Flat

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Abstract

In this paper we give some characterizations of rings over which every Gorenstein flat module is flat.

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1 Introduction

Throughout this paper R denotes a commutative ring with identity element and all modules are unital R-modules. For an R-module M, we use $\mathrm{fd}_R(M)$ to denote the usual flat dimension of M. By $\mathrm{wdim}(R)$ we denote the weak dimension of R; i.e., the supremum of the flat dimensions of all R-modules.
We say that M is *Gorenstein flat*, if there exists an exact sequence of flat left R-modules,

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots ,$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and such that $I \otimes_R -$ leaves the sequence exact whenever I is an injective right R-module.
For a positive integer n, we say that M has *Gorenstein flat dimension* at most

n, and we write $\mathrm{Gfd}_R(M) \leq n$, if M has a Gorenstein flat resolution of length n; that is an exact sequence:

$$0 \to G_n \to \cdots \to G_0 \to M \to 0,$$

where each G_i is Gorenstein flat left R-module (please see [8, 11, 13]).

The notion of Gorenstein flat modules was introduced and studied over Gorenstein rings, by Enochs, Jenda, and Torrecillas [12], as a generalization of the notion of flat modules in the sense that an R-module is flat if and only if it is Gorenstein flat with finite flat dimension. In [7], Chen and Ding generalized known characterizations of Gorenstein flat modules (then of the Gorenstein flat dimension) over Gorenstein rings to n-FC rings (coherent with finite self-FP-injective dimension). And in [13], Holm relies on the use of character modules over coherent rings to translate results for Gorenstein injective modules to the setting of Gorenstein flat modules; and so he has generalized the study of the Gorenstein flat dimension to coherent rings. Recently, Bennis enlarged the class of study of Gorenstein flat dimension to the class of the so-called GF-closed ring, such that a ring is called GF-closed if the class of Gorenstein flat modules is closed under extension [2] (see also [3]).

In this paper, we are concerned with the rings over which every Gorenstein flat module is flat. In Theorem 2.1, we characterize these kind of rings using the notion of strong Gorenstein flat modules, which is introduced in [4] as follows: a module M is said to be *strongly Gorenstein flat*, if there exists a complete flat resolution of the form

$$\mathbf{F} = \ \cdots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

such that $M \cong \mathrm{Im}(f)$.
Also we show that $\mathrm{G-wdim}(R) = \mathrm{wdim}(R)$ for a ring R over which every Gorenstein flat module is flat (Corollary 2.2). In Theorem 2.4, we prove that the propriety "all Gorenstein flat modules are flat" is local.

2 Main results

We start with the following result which gives a characterization of rings over which all Gorenstein flat R-modules are flat.

Theorem 2.1 *Let R be a ring. The following conditions are equivalent:*

1. *All Gorenstein flat R-modules are flat;*

2. *All strongly Gorenstein flat R-modules are flat;*

3. *For any R-module M,* $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious.
(2) $\Rightarrow$ (1). Follows from [4, Theorem 3.5].
(1) $\Rightarrow$ (3). Let M be an R-module. It is known that $\mathrm{Gfd}_R(M) \leq \mathrm{fd}_R(M)$. Then, it remains to prove that $\mathrm{fd}_R(M) \leq \mathrm{Gfd}_R(M)$. Obviously this inequality holds if $\mathrm{Gfd}_R(M) = \infty$. Then, assume that $\mathrm{Gfd}_R(M) = m$ for some positif integer m. Then M has a Gorenstein flat resolution of length m. Since all Gorenstein flat modules are flat, $\mathrm{fd}_R(M) \leq m = \mathrm{Gfd}_R(M)$. Therefore, $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$. ■

Next result shows that the condition “all Gorenstein flat R-modules are flat” suffices so that the classical weak global dimension and the Gorenstein weak global dimension be equal. Recall, for a ring R, that the Gorenstein weak global dimension of R is the quantity $\mathrm{G-wdim}(R) = \sup\{\mathrm{Gfd}(M) |\, M\ \textit{is an}\ R\textit{-module}\}$ (see [5]).

Corollary 2.2 *Let R be a ring satisfies the property “all Gorenstein flat R-modules are flat”, then* $\mathrm{wdim}(R) = \mathrm{G-wdim}(R)$.

Proof. This is a direct consequence of Theorem 2.1. ■

Now we show that over rings satisfying the property “all Gorenstein flat R-modules are flat”, we have every finitely generated strongly Gorenstein projective R-module is projective.

Proposition 2.3 *Let R be a ring. If every Gorenstein flat R-module is flat, then every finitely generated strongly Gorenstein projective R-module is projective.*

Proof. Let M be a finitely generated strongly Gorenstein projective module. From [4, Proposition 3.9], M is strongly Gorenstein flat and finitely presented. Then, M is flat. Therefore, from [14, Colloraly 3.58], M is projective. ■

Finally, the following results shows that the property “every Gorenstein flat R-module is flat” is a local property.

Theorem 2.4 *Let R be a ring. The following conditions are equivalent:*

1. *Every Gorenstein flat R-module is flat;*

2. *For any prime ideal p of R, every Gorenstein flat R_p-module is flat;*

3. *For any maximal ideal p of R, every Gorenstein flat R_m-module is flat.*

Proof. (1) $\Rightarrow$ (2). Let p be a prime ideal of R and let M be a strongly Gorenstein flat R_p-module. From Theorem 2.1 it remains to prove that M is projective. First we prove that M is also a strongly Gorenstein flat R-module. From [4, Definition 3.1], there exists an exact sequence of R_p-modules:

$$0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is a flat R_p-module. Since R_p is a flat R_p-module, F is then a flat R-module. Thus, it remains to prove that $\mathrm{Tor}_R(M, I) = 0$ for any injective R-module I. From [6, Proposition 4.1.1],

$$\mathrm{Tor}_R(I, M) = \mathrm{Tor}_{R_p}(Hom_R(R_p, I), M) = 0,$$

then M is a strongly Gorenstein flat R-module; and, by hypothesis, it is flat. It follows that $M = M_p$ is a flat R_p-module.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Let M be a strongly Gorenstein flat R-module. From [4, Proposition 3.6], there exists a short exact sequence of R-modules:

$$0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is a flat R-module and $\mathrm{Tor}_R(M, I) = 0$ for any injective R-module I. Let m be a maximal ideal of R. Then, the sequence of R_m-modules:

$$0 \longrightarrow M \otimes_R R_m \longrightarrow F \otimes_R R_m \longrightarrow M \otimes_R R_m \longrightarrow 0$$

is exact and $F \otimes_R R_m$ is a flat R_m-module. On the other hand, let J be an injective R_m-module, then $\mathrm{Tor}_{R_m}(M \otimes_R R_m, J) \cong \mathrm{Tor}_R(M, J) = 0$. So, from [4, Proposition 3.6], $M \otimes_R R_m$ is a strongly Gorenstein flat R_m-module, and by hypothesis, $M \otimes_R R_m$ is a flat R_m-module. Therefore, from [1, Proposition 3.10], M is a flat R-module. ■

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