International Mathematical Forum, 5, 2010, no. 38, 1887 - 1894

An Open Problem on k-Defective Colourings of Triangle-free Graphs and their Complements

M. Simanihuruk

Department of Mathematics
Bengkulu University, Indonesia
mudinsimanihuruk@yahoo.com.au

Abstract

A graph G is (m, k)-*colourable* if its vertices can be coloured with m colours such that the maximum degree of the subgraph induced on vertices receiving the same colour is at most k. *The k-defective chromatic number* $\chi_k(G)$ is the least positive integer m for which G is (m, k)-colourable. Maddox proved that $\chi_k(G) + \chi_k(\bar{G}) \leq 5\lceil \frac{p}{3k+4} \rceil$ whenever G is a triangle-free of order p and $k \geq 0$ is an integer. Simanihuruk et al proved that Maddox's upper bound is a weak upper bound for $k = 1$. In this paper a better upper bound of $\chi_k(G) + \chi_k(\bar{G})$ is established whenever G is a triangle-free graph and $k = 2$. Hence finding a sharp upper bound of $\chi_k(G) + \chi_k(\bar{G})$ is an open problem whenever G is a triangle-free graph.

Mathematics Subject Classification: 05C15, 05C35

Keywords: defective, chromatic, colouring, triangle-free, Nordhaus-Gaddum

1 Introduction

All graphs considered in this paper are finite, undirected with no loops or multiple edges. For undefined concepts and notation we refer the reader to Chartrand and Lesniak [6]. If U is a subset of the vertex set $V(G)$ of a graph G then $G[U]$ denotes the subgraph induced on U. For a vertex u of G, $d_G(u)$ denotes the degree of u and $N_G(u)$ is the set of all neigbours of u in G.

Let k be a non-negative integer. A subset U of $V(G)$ is said to be k-*independent* if the maximum degree in $G[U]$ is at most k. Note that a 0-independent set is an independent set in the usual sense. A graph G is (m, k)-*colourable* if there exists an assignment of m colours, say $1, 2, ..., m$, to the vertices of G, one colour to each vertex, such that the subgraph induced on the set of vertices that are assigned the same colour is k-independent. This type of colouring is sometimes referred to as *k-defective colouring* in the literature. Clearly any (m, k)-colouring of G produces a partition $V_1, V_2, \cdots, V_m$ of $V(G)$ such that V_i is k-independent for each i. The sets V_i are referred to as the *colour classes*. The least integer m for which G is (m, k)-colourable is called the *k-defective chromatic number* $\chi_k(G)$ of G. Note that $\chi_0(G)$ is the usual chromatic number of G. It is easy to see that $\chi_k(G) \leq \lceil \frac{p}{k+1} \rceil$, where p is the order of G.

The Nordhaus-Gaddum problem[16] associate with the parameter $f(G)$ of a graph G is to find sharp bound for $f(G) + f(\bar{G})$ and $f(G).f(\bar{G})$. Many authors have established the Nordhaus-Gaddum problem associate with the parameter $f(G)$. For examples if $f(G)$ is a *point arboricity number* $a(G)$ of graph G of order p then

$2\sqrt{p} \leq a(G) + a(\bar{G}) \leq \frac{p+3}{2}$, $p \leq a(G).a(\bar{G}) \leq (\frac{p+3}{4})^2$ (Mitchem [15]);

if $f(G)$ is a *point k-point partition number* $\rho_k(G)$ of graph G of order p then

$2\sqrt{\frac{p}{f(k)}} \leq \rho_k(G) + \rho_k(\bar{G}) \leq \frac{p+1+2k}{k+1}$,

$\frac{p}{f(k)} \leq \rho_k(G).\rho_k(\bar{G}) \leq (\frac{p+1+2k}{2(k+1)})^2$

where $f(k) = \frac{1+4\sqrt{1+8k}}{2}$ (Lick and White [12]);

if $f(G)$ is a *clique-chromatic number* $\chi_k(G, \omega)$ of graph G of order p then

$\lceil \sqrt{\frac{4p}{R-1}} \rceil \leq \chi_k(G, \omega) + \chi_k(\bar{G}, \omega) \leq \lfloor \frac{p+2k-5}{k-2} \rfloor$,

$\lceil \frac{p}{R-1} \rceil \leq \chi_k(G, \omega).\chi_k(\bar{G}, \omega) \leq \lfloor \frac{1}{2} \lfloor \frac{p+2k-5}{k-2} \rfloor \rfloor \lceil \frac{1}{2} \lfloor \frac{p+2k-5}{k-2} \rfloor \rceil$

where $R = R(n-1, n-1)$ is the Ramsey number (Achuthan [1]); and many more.

The Nordhaus-Gaddum problem is unsolved whenever $f(G)$ is the parameter *k-defective chromatic number* $\chi_k(G)$. However partial results have been established in the literature. Achuthan et al [2] proved that $\chi_k(G).\chi_k(\bar{G}) \geq \frac{p}{R-1}$ where $R = 2k + 1$ for k is odd or $R = 2k$, otherwise. The determination of a sharp upper bound of $\chi_k(G) + \chi_k(\bar{G})$ is an open problem. Some researchers have investigated this problem.

Maddox [14] proved $\chi_k(G) + \chi_k(\bar{G}) \leq 5\lceil \frac{p}{3k+4} \rceil$ if G is a triangle-free graph of order p. When $k = 1$ he improved the above bound to $6\lceil \frac{p}{9} \rceil$. Furthermore he suggested the following conjecture for $k \geq 1$: For a graph G of order p, $\chi_k(G) + \chi_k(\bar{G}) \leq 2 + \lceil \frac{p-1}{k+1} \rceil$. This conjecture is true whenever G is triangle-free graph and $k = 1$(Simanihuruk et al [18]). Hence the upper bound of Maddox is a weak upper bound for $k = 1$. Therefore finding a sharp upper bound of $\chi_k(G) + \chi_k(\bar{G})$ is an open problem whenever G is a triangle-free graph.

In [2] Achuthan et al proved that Maddox's conjecture is also true whenever G is a P_4-free graph and $k = 1$. Further they disproved Maddox's conjecture for all $k \geq 1$ by constructing a graph G of order $p \equiv 1(mod(k + 1))$ with $\chi_k(G) + \chi_k(\bar{G}) = 3 + \lceil \frac{p-1}{k+1} \rceil$. Achuthan et al [2] also established the following weak upper bound: For a graph G of order p, $\chi_k(G) + \chi_k(\bar{G}) \leq \frac{2p+2k+4}{k+2}$. For k = 1, Achuthan et al [3] proved that $\chi_1(G) + \chi_1(\bar{G}) \leq \lceil \frac{2p+4}{3} \rceil$ for any graph G of order p.

In this paper we will study the Nordhaus-Gaddum problem for 2-defective chromatic number over the class of triangle-free graphs. We will prove that $\chi_2(G) + \chi_2(\bar{G}) \leq 2 + \lceil \frac{p+3}{3} \rceil$ whenever G is a triangle-free graph of order p. The proof depend on the characterization of the smallest order of triangle-free graph G with $\chi_2(G) = 3$ which was established in [4].

We define $f(m, k)$ to be the smallest order of a triangle-free graph G with $\chi_k(G) = m$. Clearly $f(2, k) = k + 2$. The problem of determining $f(m, k)$ is unsolved even for $k = 0$ (see Toft [19]). It is easy to see that $f(3, 0) = 5$. Chvátal [7] has shown that $f(4, 0) = 11$. Jensen and Royle [11] have shown that $f(5, 0) = 22$. We refer the reader to Avis [5], Hanson and MacGillivray [9] and Grinstead et al [8] for related results. Simanihuruk et al [18] proved that $f(3, 1) = 9$ and also completely determined the class of triangle-free graphs of order 9 with $\chi_1(G) = 3$. Similarly Achuthan et al[4] proved that f(3,2) = 13 and also completely determined the class of triangle-free graphs of order 13 with $\chi_2(G)$ =13.

In all the figures of this paper, a double line (a double dotted line) between sets X and Y means that every (no) vertex of X is adjacent to every (any) vertex of Y. Similarly a line (dotted line) between two vertices sets x and y means that the edge xy is (is not) in the graph.

2 Preliminary Notes

This section provide some previous result that will be used to prove the main results of this paper.

Hopkins and Staton [9] and Lovász [13] generalized Brook's type result in the following theorem.

Theorem 2.1 *([9]and [13]) For a graph G with maximum degree Δ, we have* $\chi_k(G) \leq \lceil \frac{\Delta+1}{k+1} \rceil$.

The following three theorems were established in Achuthan et al [4]

Theorem 2.2 *[4] For integers $k \geq 0$ and $m \geq 3$,* $f(m, k) \geq (k + 1) \binom{m}{2} + m - 1$.

Theorem 2.3 *[4] The smallest order of a triangle-free graph G with $\chi_2(G) = 3$ is 13, that is,* $f(3, 2) = 13$

The proof of the main results of this paper depend on the following results.

Theorem 2.4 *[4] Let G be a triangle-free graph of order 13. Then $\chi_2(G) = 3$ if and only if G is isomorphic to one of the graphs $G_i, 1 \leq i \leq 3$ shown in Figure 1. In these figures, $G_i[A_j] \cong \overline{K_3}$ for $1 \leq i \leq 3$ and $1 \leq j \leq 4$.*

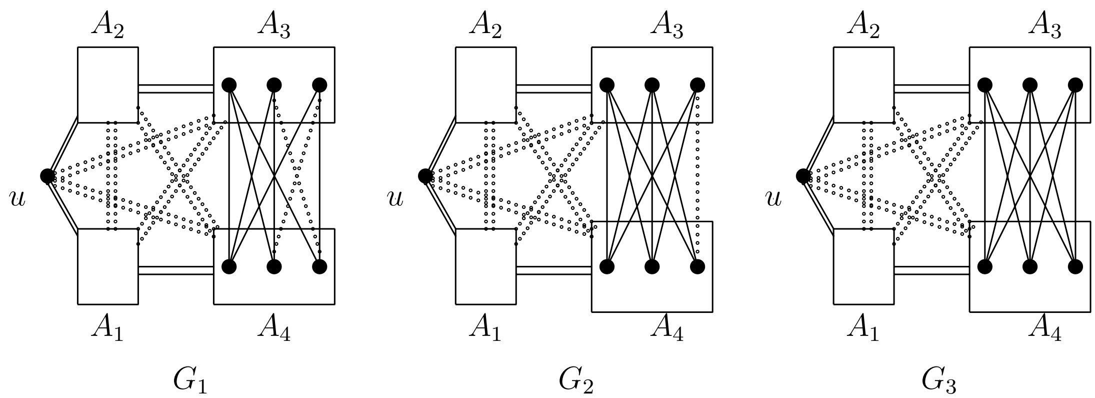

Figure 1: Graphs G_1, G_2 and G_3

Maddox [14] proved the upper bound of the sum $\chi_k(G) + \chi_k(\bar{G})$ for a class of triangle-free graph in the following theorem.

Theorem 2.5 *[14] If G is triangle-free graph of order p then $\chi_k(G)+\chi_k(\bar{G}) \leq 5\lceil \frac{p}{3k+4} \rceil$.*

Simanihuruk et al [18] proved that the upper bound of Theorem 2.5 is a weak upper bound for $k = 1$. In this note we will also show that the upper bound of Theorem 2.5 is a weak upper bound for $k = 2$. Hence the determination of the sharp upper bound of the sum $\chi_k(G) + \chi_k(\bar{G})$ for a class of triangle-free graph is an open problem.

3 Main Results

The results of this paper depend on the improvement of the lower bound of Theorem 2.2 for k = 2.

Lemma 3.1 *For integer $m \geq 4$, $f(m, 2) \geq \frac{(3m+8)(m-3)}{2} + 13$.*

Proof: Consider a triangle-free graph G of order $f(m, 2)$ such that $\chi_2(G) = m$. Let u be a vertex of degree $\Delta(G)$, $A = N_G(u)$, $B = V(G) - A - \{u\}$ and $G[B] = H$. Since G is triangle-free, A is independent. Also the order of H is at least $f(m - 1, 2)$, for otherwise, H is $(m - 2, 2)$-colourable. This implies that G is $(m-1,2)$-colourable, a contradiction to the assumption that $\chi_2(G) = m$. Thus the order of G is $f(m,2) \geq \Delta(G)+1+f(m-1,2)$. Now using Theorem 2.1, it is easy to show that $\Delta(G) \geq 3(m-1)$. Thus

$$\begin{aligned} f(m,2) &\geq \Delta(G)+1+f(m-1,2) \\ &\geq 3(m-1)+1+f(m-1,2). \end{aligned}$$

Proceeding in this manner we have

$$\begin{aligned} f(m,2) &\geq 3[(m-1)+(m-2)+...+3]+(m-3)+f(3,2) \\ &\geq \frac{(3m+8)(m-3)}{2}+f(3,2). \end{aligned}$$

Now applying Theorem 2.3 in the last inequality we have the required inequality. This completes the proof of the lemma.

Next we will state the main result of this paper.

Theorem 3.1 *Let G be a triangle-free graph of order p. Then* $\chi_2(G)+\chi_2(\bar{G}) \leq 2+\lceil\frac{p+3}{3}\rceil$

Proof: First let $\chi_2(G) \leq 2$. If $\chi_2(G)=1$ then $\chi_2(\bar{G}) \leq \lceil\frac{p}{3}\rceil$. Hence $\chi_2(G)+\chi_2(\bar{G}) \leq 2+\lceil\frac{p+3}{3}\rceil$. If $\chi_2(G)=2$ then G has $K(1,3)$. The vertices of $K(1,3)$ is 2-independent in $\bar{G}$ and therefore $\chi_2(\bar{G}) \leq \lceil\frac{p-4}{3}\rceil+1 = \lceil\frac{p-1}{3}\rceil$. Hence $\chi_2(G)+\chi_2(\bar{G}) \leq 2+\lceil\frac{p+3}{3}\rceil$.

From now on we assume $\chi_2(G) \geq 3$. We prove the theorem by induction on p. Let G be a triangle-free graph of order 13. By Theorem 2.4 we have $\chi_2(G) \leq 3$. Thus $\chi_2(G)=3$. Now by Theorem 2.4, G is isomorphic to one of the graph G_i, $1 \leq i \leq 3$ of Theorem 4. It is easy to see that $\chi_2(\bar{G}_i) \leq 3$. Hence $\chi_2(\bar{G}) \leq 3$. Therefore $\chi_2(G)+\chi_2(\bar{G}) \leq 2+\lceil\frac{p+3}{3}\rceil$.

Next let $p \geq 14$. We make the induction hypothesis that the theorem is true for every triangle-free graph of order less than p and then prove it for any triangle-free graph of order p.

Case 1: There is a subset L of cardinality 13 of $V(G)$ such that $\chi_2(G[L]) = 3$.

By Theorem 2.4, $G[L]$ is isomorphic to one of the graph Theorem 2.4. Each G_i , $1 \leq i \leq 3$ has a subgraph $H = G_1[A_1 \cup A_2 \cup A_3 \cup A_4]$. By Theorem 2.3 and minimality of $f(3,2)$, we have that $\chi_2(H) \leq 2$, since the number of vertices of H is 12. It is also easy to verify that $\chi_2(\bar{H}) \leq 2$. Hence $\chi_2(G)+\chi_2(\bar{G}) \leq \chi_2(G-H)+\chi_2(\bar{G}-H)+\chi_2(H)+\chi_2(\bar{H}) \leq 2+\lceil\frac{p-12+3}{3}+\rceil+2+2 \leq 2+\lceil\frac{p+3}{3}\rceil$. This proves the theorem in this case.

Case 2: For every subset L of cardinality 13 of $V(G)$ we have $\chi_2(G[L]) \leq 2$.

Since $\chi_2(G) \geq 3$, G contains a $K(1,3)$. Let t be the largest number of vertex disjoint $K(1,3)$ in G and $Q_1, Q_2, ..., Q_t$ are the t vertex disjoint $K(1,3)$ in G. Let $M = \bigcup_{i=1}^{t} V(Q_i)$. Note that $V(G) - M$ is 2-independent in G and the subgraph $\bar{G}[V(Q_i)]$ is $K(1,3)$-free for each i. Thus

$$\chi_2(\bar{G}[M]) \leq t \tag{1}$$

Since $\bar{G} - M$ is a graph of order $p - 4t$, we have

$$\chi_2(\bar{G}) \leq \chi_2(\bar{G}[M]) + \chi_2(\bar{G} - M) \leq t + \lceil \frac{p-4t}{3} \rceil = \lceil \frac{p-t}{3} \rceil \tag{2}$$

Also

$$\chi_2(G) \leq \chi_2(G[M]) + \chi_2(G - M) \leq \chi_2(G[M]) + 1 \tag{3}$$

First let $t \geq 24$ and let $N = \bigcup_{i=1}^{24} V(Q_i)$. Note that $|N| = 96$. Using Lemma 3.1 we have $f(9,2) \geq 118$. Therefore from the minimality of $f(9,2)$ and the fact that $|N| = 96$, we have $\chi_2(G[N]) \leq 8$. Since $V(Q_i)$ is 2-independent in $\bar{G}$ for each i, $1 \leq i \leq 24$, it follows that $\chi_2(\bar{G}[N]) \leq 24$. Now $\chi_2(G) \leq \chi_2(G[N]) + \chi_2(G - N)$ and $\chi_2(\bar{G}) \leq \chi_2(\bar{G}[N]) + \chi_2(\bar{G} - N)$. Thus $\chi_2(G) + \chi_2(\bar{G}) \leq \chi_2(G[N]) + \chi_2(\bar{G}[N]) + \chi_2(G - N) + \chi_2(\bar{G} - N) \leq 8 + 24 + \chi_2(G - N) + \chi_2(\bar{G} - N) \leq 32 + \chi_2(G - N) + \chi_2(\bar{G} - N)$. By induction hypothesis, $\chi_2(G - N) + \chi_2(\bar{G} - N) \leq \lceil \frac{p-4.24+3}{3} \rceil + 2$. Thus $\chi_2(G) + \chi_2(\bar{G}) \leq 32 + \lceil \frac{p-4.24+3}{3} \rceil + 2 = 2 + \lceil \frac{p+3}{3} \rceil$. This prove the theorem when $t \geq 24$. Next we will consider the case $1 \leq t \leq 23$. From (2) and (3) we have

$$\chi_2(G) + \chi_2(\bar{G}) \leq \chi_2(G[M]) + 1 + \lceil \frac{p-t}{3} \rceil \tag{4}$$

By Lemma 3.1 we have $f(4,2) \geq 23, f(5,2) \geq 36, f(6,2) \geq 52, f(7,2) \geq 71$ and $f(8,2) \geq 93, f(9,2) \geq 118$ and by Theorem 2.3 we have $f(3,2) = 13$.

Notice that $|M| = 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92$ respectively for $t = 1, 2, ..., 23$. Using these facts and the minimality of $f(3,2), f(4,2), f(5,2), f(6,2)$ and $f(7,2)$ we have

$$\chi_2(G[M]) \leq \begin{cases} 2 & \text{if } t = 1, 2, 3; \\ 3 & \text{if } t = 4, 5; \\ 4 & \text{if } t = 6, 7, 8; \\ 5 & \text{if } t = 9, 10, 11, 12; \\ 6 & \text{if } t = 13, 14, ..., 17; \\ 7 & \text{if } t = 18, 19, ..., 23. \end{cases}$$

Combining these last inequality with inequality (4) we have the required inequality for each t = 1,2,...,23. This completes the proof of the theorem.

Remark: For $k = 2$ and $p \geq 11$ it is not difficult to show that $2 + \lceil \frac{p+3}{3} \rceil \leq 5\lceil \frac{p}{10} \rceil$. Hence the upper bound of Theorem 3.1 is better than that of Theorem 2.5 for $k = 2$ and $p \geq 11$.

ACKNOWLEDGEMENTS. The author thank the Directorate General Higher Education, Department of National Education, Republic of Indonesia, for providing the 2007 Fundamental Research Award Contract No.009/SP2H/ PP/DP2M/III/2008.

References

[1] N.Achuthan, On the n-clique chromatic number of complementary graphs, *Annals of Discrete Mathematics,* **9**(1980), 285-290.

[2] N.Achuthan, N.R.Achuthan and M.Simanihuruk, On defective colourings of complementary graphs, *The Australasian Journal of Combinatorics,* **13**(1996), 175-196.

[3] N.Achuthan, N.R.Achuthan and M.Simanihuruk, On the Nordhaus-Gaddum problem for the n-path chromatic number of a graph, *J. Combinatorial Mathematics and Combinatorial Computing,* **27**(1998), 107-115.

[4] N.Achuthan, N.R.Achuthan and M.Simanihuruk. On minimal triangle-free graphs with prescribed k-defective chromatic number, submitted to *Discrete mathematics,* (2009).

[5] D.Avis, On minimal 5-chromatic triangle free graphs, *Journal of Graph Theory,* **3**(1987), 397-400.

[6] G.Chartrand and L.Lesniak, *Graphs and Digraphs,* 2nd Edition , Wadsworth and Brooks/Cole, Monterey California, U.S.A,1986.

[7] V.Chvátal, The minimality of the Mycielski graph, *Graphs and Combinatorics, Springer-Verlag, Berlin(Lecture Notes in Mathematics),* **406**(1973), 243-246.

[8] C.M Grinstead, M. Katinsky and D. Van Stone, On minimal triangle-free-5-chromatic graphs, *Journal Combinatorial Mathematics and Combinatorial Computing,* **6**(1989), 189-193.

[9] D.Hanson and G.MacGillivray, On small triangle-free graphs, *Ars Combinatoria,* **35**(1993), 257-263.

[10] G.Hopkins and W.Staton, Vertex partitions and k-small subsets of graphs, *Ars Combinatoria,* **22**(1986), 19-24.

[11] T.Jensen and G.F.Royle, Small graphs with chromatic number 5: A computer search, *Journal of Graph Theory,* **19**(1995), 107-116.

[12] D.R.Lick and A.T.White, Point partition numbers of complementary graphs, *Mathematica Japonicae,* **19**(1974), 233-237.

[13] L.Lovász, On the compositions of graphs, *Studia Scientiarum Mathematicarum Hungarica,* **1**(1966), 237-238.

[14] R.B Maddox, Vertex partitions and transition parameters, *Ph.D Thesis,* The University of Mississippi, Missisippi, 1988.

[15] J.Mitchem, On the point-arboricity of a graphs and its complement, *Can.J.Math.,* **23(2)**(1956), 287 -292.

[16] E.A.Nordhaus and J.W Gaddum, On complementary graphs, *American Mathematical Monthly,* **63**(1956), 175 - 177.

[17] M.Simanihuruk, N.Achuthan and N.R.Achuthan, On minimal triangle-free graphs with prescribed 1-defective chromatic number, *The Australasian Journal of Combinatorics,* **16**(1997), 203-227.

[18] M.Simanihuruk, N.Achuthan and N.R Achuthan, On defective colourings of triangle-free graphs, *The Australasian Journal of Combinatorics,* **16**(1997), 259-283.

[19] B.Toft, 75 graph-colouring problems, *Graph Colourings* (Nelson,R and Wilson,R.J. eds.), Longman Scientific Technical, England,1990, 9-35.

Received: January, 2010