International Mathematical Forum, 5, 2010, no. 31, 1547 - 1549

Another Proof of Ewell's Octuple Product Identity via Liouville's Theorem

Qinglun Yan [1] **and Xiaona Fan**

College of Science, Nanjing University of Posts and Telecommunications
Nanjing 210046, P.R. China
yanqinglun@yahoo.com.cn

Abstract. In terms of Liouville's Theorem, we give a new proof of Ewell's octuple product identity.

Mathematics Subject Classification: 05A30, 14K25

Keywords: Octuple product identity; Jacobi's triple product identity

For two complex a and q, the shifted-factorial of a with base q is defined by

$$(a;q)_0 = 1 \quad \text{and} \quad (a;q)_m = \prod_{n=0}^{m-1}(1-aq^n) \quad \text{for} \quad m = 1, 2, \cdots .$$

When $|q| < 1$, the following products of infinite order are well defined:

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

For the sake of brevity, the product of shifted factorials is abbreviated to

$$[\alpha, \beta, \cdots, \gamma; q]_\infty = (\alpha;q)_\infty(\beta;q)_\infty \cdots (\gamma;q)_\infty.$$

First, we give two lemmas which will be used later without further explanation.

Lemma 1 (Jacobi's triple product identity [5, §1.6])**.**

$$[q, a, q/a; q]_\infty \; = \; \sum_{n=-\infty}^{+\infty} (-1)^n \, q^{\binom{n}{2}} \, a^n.$$

Lemma 2 (Liouville's Theorem)**.** *Every bounded entire function is constant.*

[1]This work has been supported by the Foundation of NJUPT under Grant No. NY208068 and NY208069 and the Natural Sciences Foundation for Colleges and Universities in Jiangsu Province of China under Grant No. 09KJD110004.

Liouville's Theorem is fundamental in complex analysis, which can be utilized to prove theta function identities [2, 3, 6] etc.. Here, combining the Log-Derivative method, we shall give a new proof of Ewell's octuple product identity, which supplies expressions of $(q;q)_\infty^6(q;q^2)_\infty^2$ and $(q;q)_\infty^8$ [1, 4].

Theorem 3 (Ewell [4])**.** *For $a \neq 0$ and $|q| < 1$, there holds*

$$\begin{aligned} &[q,q,a,qa,1/a,q/a;q]_\infty \left[a^2q, q/a^2; q^2\right]_\infty \\ &= 2P(q) \sum_{n=-\infty}^{\infty} (-1)^n a^{4n} q^{2n^2} - Q(q) \sum_{n=-\infty}^{\infty} (-1)^n a^{4n} q^{2n^2} (aq^n + a^{-1}q^{-n}), \end{aligned}$$

where $P(q) = (q^4;q^4)_\infty$ and $Q(q) = [q^{12}, q^5, q^7; q^{12}]_\infty + q\,[q^{12}, q, q^{11}; q^{12}]_\infty$.

This octuple product identity was first discovered by Ewell [4] in 1982. In 2004, Chen-Chen-Huang [1] offered two constructive proofs.

Proof. In terms of the multisection method for triple product and Jacobi's triple product identity, $Q(q)$ can be reformulated as follows

$$Q(q) = \sum_{n=-\infty}^{\infty} (-1)^n (-q)^{\frac{3n^2-n}{2}} = (-q;-q)_\infty = \left[-q, q^2; q^2\right]_\infty.$$

The equation in Theorem 3 can be restated as $F(a) = G(a)/H(a) = 1$, where

$$\begin{aligned} G(a) &= 2P(q) \sum_{n=-\infty}^{\infty} (-1)^n a^{4n} q^{2n^2} - Q(q) \sum_{n=-\infty}^{\infty} (-1)^n a^{4n} q^{2n^2} (aq^n + a^{-1}q^{-n}), \\ H(a) &= [q,q,a,qa,1/a,q/a;q]_\infty \left[a^2q, q/a^2; q^2\right]_\infty. \end{aligned}$$

Then it is trivial to see that $G(a) = -a^4q^2G(aq)$ and $H(a) = -a^4q^2H(aq)$, which follows that $F(a) = F(aq) = F(aq^2) = \cdots$.

On the other hand, possible poles of $F(a)$ are given by zeros of $H(a)$, which consist of $a = q^n, \pm q^{n+\frac{1}{2}}$ with $n \in \mathbb{Z}$. However, $G(q^n) = G(\pm q^{n+\frac{1}{2}}) = 0$ for $n \in \mathbb{Z}$, which are justified by $G(1) = G(\pm q^{\frac{1}{2}}) = 0$.

Noting that $a = q^n$ is a double zero of $H(a)$, we need to check $G'(q^n) = 0$. Since the function $G(a)$ satisfies $G(a) = -a^4q^2G(aq)$, to prove that $G'(q^n) = 0$ it suffices to prove that $G'(1) = 0$. Using the Log-Derivative method, we have

$G'(a) = I_1(a) + I_2(a)$, where

$$I_1(a) = 8\left[q^4, q^4, a^4q^2, q^2/a^4; q^4\right]_\infty \sum_{n=0}^{+\infty} \left\{ \frac{q^{4n+2}/a^5}{1-q^{4n+2}/a^4} - \frac{q^{4n+2}a^3}{1-q^{4n+2}a^4} \right\},$$

$$I_2(a) = 4a\left[-q, q^2; q^2\right]_\infty \left[q^4, a^4q^3, q/a^4; q^4\right]_\infty \sum_{n=0}^{+\infty} \left\{ \frac{q^{4n+1}/a^5}{1-q^{4n+1}/a^4} - \frac{a^3q^{4n+3}}{1-a^4q^{4n+3}} \right\}$$

$$+ \frac{4}{a}\left[-q, q^2; q^2\right]_\infty \left[q^4, a^4q, q^3/a^4; q^4\right]_\infty \sum_{n=0}^{+\infty} \left\{ \frac{q^{4n+3}/a^5}{1-q^{4n+3}/a^4} - \frac{a^3q^{4n+1}}{1-a^4q^{4n+1}} \right\}$$

$$+\left[-q, q^2; q^2\right]_\infty \left[q^4, a^4q^3, q/a^4; q^4\right]_\infty - \frac{1}{a^2}\left[-q, q^2; q^2\right]_\infty \left[q^4, a^4q, q^3/a^4; q^4\right]_\infty.$$

After some computation, we derive $I_1(1) = I_2(1) = 0$, which leads to $G'(1) = 0$, i.e. $G'(q^n) = 0$, that is to say, $a = q^n$ is also a double zero of $G(a)$.

Therefore, $F(a)$ is a holomorphic function on the whole complex plane and must be constant thanks to Liouville's Theorem. All that remains is to show that this constant is one. Of course, we can get it for an appropriately chosen value of a. For example, taking $a = -1$, we see that

$$F(a) = F(-1) = G(-1)/H(-1) = 1$$

due to $G(-1) = 4(q^2; q^2)^2_\infty = 2(q^2; q^2)^2_\infty + 2(q^2; q^2)^2_\infty = H(-1)$, which leads us to Ewell's octuple product identity in Theorem 3. □

References

[1] Sin-Da Chen, Wei-Yueh Chen and Sen-Shan Huang, A new construcation of Ewell's octuple product identity, Indian J. Pure Appl. Math. 35 (2004), 1241-1253.

[2] W. Chu and Q. Yan, Verification method for theta function identities via Liouvilles theorem, Arch. Math. 90 (2008), 331-340.

[3] R. J. Evans, Theta function identities, J. Math. Anal. Appl. 147 (1990), 97–121.

[4] J. A. Ewell, On an octuple-product identity, Rocky Mountain J. Math. 12 (1982), 279-282.

[5] G. Gasper and M. Rahman, Basic Hypergeometric Series (2nd edition), Cambridge University Press, 2004.

[6] Q. Yan, A new proof of the septuple product identity, Dis. Math. 309 (2009), 2589-2591.

Received: November, 2009