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Several Explicit Evaluations for Ratios of Ramanujan's Theta Function $\phi(q)$

Qinglun Yan [1] **and Xiaona Fan**

College of Science, Nanjing University of Posts and Telecommunications
Nanjing 210046, P.R. China
yanqinglun@yahoo.com.cn

Abstract. In this paper we first give alternative proofs of two Ramanujan's theta function identities. Then we derive several explicit evaluations for ratios of Ramanujan's theta function $\phi(q)$.

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1. Introdution

Throughout the paper, we always assume that $|q| < 1$. For positive integer n, we employ the standard notation

$$(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i) \qquad \text{and} \qquad (a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i).$$

Ramanujan's general theta function is defined by

$$f(a,b) = \sum_{n=-\infty}^{+\infty} a^{n(n+1)/2} b^{n(n-1)/2},$$

where $|ab| < 1$. If we set $a = qe^{2iz}$, $b = qe^{-2iz}$ and $q = e^{\pi i\tau}$, where z is a complex number and $Im(\tau) > 0$, then $f(a,b)$ is the classical theta function $\vartheta_3(z,\tau)$ [11, P. 464].

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The three most important special cases of $f(a,b)$ are

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{+\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty,$$

$$\phi(q) := f(q,q) = 1 + 2\sum_{n=1}^{+\infty} q^{n^2},$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{+\infty} q^{n(n+1)/2}$$

We shall frequently appeal, without further explanation, to the celebrated Jacobi triple product identity [3, Entry 19, p. 35]:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

In the unorganized pages of his notebooks, Ramanujan recorded many beautiful identities. For example,

$$\text{(1.1a)} \qquad 1 - \frac{\chi^3(-q)}{\chi(-q^3)} = 3q\frac{\psi(q^9)}{\psi(q)}, \qquad 1 - \frac{\chi^5(-q)}{\chi(-q^5)} = 5q\frac{\psi^2(q^5)}{\psi^2(q)},$$

$$\text{(1.1b)} \qquad 1 + 2\frac{\chi(-q^3)}{\chi^3(-q)} = 3\frac{\phi(-q^9)}{\phi(-q)}, \qquad 1 + 4\frac{\chi(-q^5)}{\chi^5(-q)} = 5\frac{\phi^2(-q^5)}{\phi^2(-q)};$$

where $\chi(q) := (-q;q^2)_\infty$, only for notational purposes. Berndt [4, 5] proved theses identities via parameterization. Different proofs can be found in Kang [7] and Kongsiriwong-Liu [8] etc.. Using some other identities of theta functions, Baruah-Bhattracharyya [2] proved the identities (1.1a) and used them to deduce some theorems for the explicit evaluation of Ramanujan's theta function $\psi(x)$ in terms of Weber-Ramanujan class invariants. In this paper, we shall first prove the other two identities (1.1b) in Section 2, and then derive several explicit evaluations for ratios of Ramanujan's theta function $\phi(q)$ in Section 3.

2. Two Ramanujan's theta function identities

Theorem 1.

$$1 + 2\frac{\chi(-q^3)}{\chi^3(-q)} = 3\frac{\phi(-q^9)}{\phi(-q)}.$$

Proof. By means of [3, Corollary (ii) of Entry 31, p. 49], we find $\psi(q) - q\psi(q^9) = f(q^3)$. Using $f(q) = \frac{f(-q^2)f^2(-q^3)}{f(-q)f(-q^6)} = \frac{\phi(-q^3)}{\chi(-q)}$ in the last identity and simplifying, we have

$$\text{(2.1)} \qquad \psi(-q)\chi(q^3) - \phi(q^9) = -q\psi(-q^9)\chi(q^3).$$

Recalling [3, Corollary (i) of Entry 31, p. 49 and Example (v), p. 51], we find

$$\phi(q)-\phi(q^9)=2q\psi(-q^9)\chi(q^3). \tag{2.2}$$

Adding two times of (2.1) to (2.2), we have $\phi(q)-3\phi(q^9)=-2\psi(-q)\chi(q^3)$. Noting $\frac{\phi(q)}{\psi(-q)}=\frac{\chi^2(q)f(-q^2)}{\chi(-q)f(-q^4)}=\chi^3(q)$ and simplifying, we derive the very identity appeared in the theorem. □

Theorem 2.

$$1+4\frac{\chi(-q^5)}{\chi^5(-q)}=5\frac{\phi^2(-q^5)}{\phi^2(-q)}.$$

Proof. By means of [3, Entry 9(vii), p. 258 and Entry 10(v), p. 262], we find

$$\psi^2(q)-q\psi^2(q^5)=\frac{\phi(-q^5)f(-q^5)}{\chi(-q)}.$$

Noting $f(q)=\frac{\phi(q)}{\chi(q)}$, we deduce $\psi^2(q)-q\psi^2(q^5)=\frac{\phi^2(-q^5)}{\chi(-q)\chi(-q^5)}$. With the help of $\psi^2(q^5)=\frac{f(q^5)f(-q^{10})f(-q^{20})}{f(-q^5)}=\frac{f(q^5)f(-q^{20})}{\chi(-q^5)}$, we get

$$\psi^2(q)-q\frac{f(q^5)f(-q^{20})}{\chi(-q^5)}=\frac{\phi^2(-q^5)}{\chi(-q)\chi(-q^5)},$$

that's to say,

$$\phi^2(-q^5)=\psi^2(q)\chi(-q)\chi(-q^5)-q\chi(-q)f(q^5)f(-q^{20}). \tag{2.3}$$

Now, we recall from [3, Entry 9(iii), p. 258] that

$$\phi^2(-q^5)-\phi^2(-q)=4q\chi(-q)f(q^5)f(-q^{20}). \tag{2.4}$$

Combing (2.3) with (2.4), we have $5\phi^2(-q^5)-\phi^2(-q)=4\psi^2(q)\chi(-q)\chi(-q^5)$. Noting $\frac{\phi(q)}{\psi(-q)}=\frac{\chi^2(q)f(-q^2)}{\chi(-q)f(-q^4)}=\chi^3(q)$ and simplifying, we prove the theorem. □

3. Explicit evaluations for ratios of Ramanujan's theta function $\phi(q)$

For $q=exp(-\pi\sqrt{n})$, where n is a positive rational number, Weber-Ramanujan class invariants G_n and g_n [5, 10] are defined by

$$G_n:=2^{-1/4}q^{-1/24}\chi(q), \qquad g_n:=2^{-1/4}q^{-1/24}\chi(-q).$$

Noting the definitions of G_n, g_n and the identities in Theorem 1 and 2, respectively, we derive two theorems.

Theorem 3 ([5, (5.7), p. 334] for the first identity)**.**

$$\frac{\phi(e^{-9\pi\sqrt{n}})}{\phi(e^{-\pi\sqrt{n}})}=\frac{1}{3}\left(1+\sqrt{2}\frac{G_{9n}}{G_n^3}\right);$$

$$\frac{\phi(-e^{-9\pi\sqrt{n}})}{\phi(-e^{-\pi\sqrt{n}})}=\frac{1}{3}\left(1+\sqrt{2}\frac{g_{9n}}{g_n^3}\right).$$

Theorem 4 ([5, (8.11), p. 339] for the first identity)**.**

$$\frac{\phi^2(e^{-5\pi\sqrt{n}})}{\phi^2(e^{-\pi\sqrt{n}})} = \frac{1}{5}\left(1+2\frac{G_{25n}}{G_n^5}\right);$$
$$\frac{\phi^2(-e^{-5\pi\sqrt{n}})}{\phi^2(-e^{-\pi\sqrt{n}})} = \frac{1}{5}\left(1+2\frac{g_{25n}}{g_n^5}\right).$$

By means of the four identities in the last two theorems and the values of G_n and g_n, we can derive several explicit values for ratios of theta function $\phi(q)$ easily. We take some values as examples.

When $n=\frac{1}{9}$, $G_1=1$, $G_{\frac{1}{9}}=\left((1+\sqrt{3})/\sqrt{2}\right)^{\frac{1}{3}}$,

Example 5. [1, Theorem 5.5, viii]

$$\frac{\phi(e^{-3\pi})}{\phi(e^{-\pi/3})}=\frac{\sqrt{3}}{3}.$$

When $n=\frac{1}{3}$, $G_3=G_{\frac{1}{3}}=2^{\frac{1}{12}}$,

Example 6. [1, Theorem 5.5, xii]

$$\frac{\phi(e^{-3\sqrt{3}\pi})}{\phi(e^{-\pi/\sqrt{3}})}=\frac{1+2^{\frac{1}{3}}}{3}.$$

When $n=2/9$, $g_2=1$, $g_{\frac{2}{9}}=(\sqrt{3}-\sqrt{2})^{\frac{1}{3}}$,

Example 7.

$$\frac{\phi(-e^{-3\sqrt{2}\pi})}{\phi(-e^{-\sqrt{2}\pi/3})}=\frac{1}{3}(3+\sqrt{6}).$$

When $n=1$, $G_1=1$, $G_{25}=G_{1/25}=(1+\sqrt{5})/2$,

Example 8. [6, Theorem 1]

$$\frac{\phi(e^{-5\pi})}{\phi(e^{-\pi})}=\frac{1}{\sqrt{5\sqrt{5}-10}}, \qquad \frac{\phi(e^{-\pi})}{\phi(e^{-\pi/5})}=\sqrt{\sqrt{5}-2}.$$

When $n=1/5$, $G_5=G_{1/5}=((1+\sqrt{5})/2)^{\frac{1}{4}}$,

Example 9. [1, Theorem 5.5(i)]

$$\frac{\phi(e^{-\sqrt{5}\pi})}{\phi(e^{-\pi/\sqrt{5}})}=5^{-\frac{1}{4}}.$$

When $n=2/5$, $g_{10}=\sqrt{(1+\sqrt{5})/2}$, $g_{2/5}=\sqrt{(-1+\sqrt{5})/2}$,

Example 10. [9, Theorem 2.6]

$$\frac{\phi(-e^{-\sqrt{10}\pi})}{\phi(-e^{-\pi\sqrt{2/5}})}=\sqrt{\frac{5+2\sqrt{5}}{5}}.$$

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