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Difference Equation on Quintuple Products and Ramanujan's Partition Congruence $p(11n + 6) \equiv 0 \pmod{11}$

Qinglun Yan [1], **Xiaona Fan and Jing Fu**

College of Science, Nanjing University of Posts and Telecommunications
Nanjing 210046, P.R. China
yanqinglun@yahoo.com.cn

Abstract. In this paper, by means of the double functional equation method, we recover a difference equation on quintuple products, and then give a new proof of the Ramanujan congruence on partition function modulo 11.

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For two complex q and x, the shifted-factorial of x with base q is defined by

$$(x;q)_\infty = \prod_{n=0}^{\infty}(1-q^n x),$$

whose product form is abbreviated as

$$[\alpha,\beta,\cdots,\gamma;q]_\infty = (\alpha;q)_\infty(\beta;q)_\infty\cdots(\gamma;q)_\infty.$$

Throughout the paper, we shall frequently appeal, without further explanation, to the celebrated quintuple product identity [4]:

$$\begin{aligned}[q,x,q/x;q]_\infty\left[qx^2,q/x^2;q^2\right]_\infty &= \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}}(q^2/x^3)^n\{1-x^{6n+1}\} \\ &= \sum_{n=-\infty}^{+\infty} q^{3\binom{n}{2}}(qx^3)^n\{1-(q/x^2)^{3n+1}\}.\end{aligned}$$

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In this paper, by means of the double functional equation method, we recover a difference equation on quintuple products, and then give a new proof of the Ramanujan congruence on partition function modulo 11.

1. Difference equation on quintuple products

Theorem 1. *For the bivariate function $f(x,y)$ defined by*

$$f(x,y) = y\big[q^2, qx^2, q/x^2; q^2\big]_\infty \big[x^4, q^4/x^4; q^4\big]_\infty \big[q^2, y^2, q^2/y^2; q^2\big]_\infty \big[q^2y^4, q^2/y^4; q^4\big]_\infty,$$

there holds the following identity on symmetric difference:

$$f(x,y) - f(y,x) = y(q;q)_\infty^2 \left[x^2, q/x^2, y^2, q/y^2, xy, q/xy, x/y, qy/x; q\right]_\infty.$$

This is one of the very important theta function identities, which is similar to Winquist's identity [6]. For more proofs, the reader can refer to [1, 2, 3] etc..

Proof. For the bivariate function $g(x,y)$ defined by the following infinite products $g(x,y) = (q;q)_\infty^2 \left[x^2, q/x^2, y^2, q/y^2, xy, q/xy, x/y, qy/x; q\right]_\infty$, it's trivial to see $g(x,y)$ is analytic on $0 < |x|, |y| < \infty$. Then we can expand it as a double Laurent series $g(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{C}_{m,n} x^m y^n$.

From the definition of $g(x,y)$, we find two functional equations $g(x,y) = qx^6 g(qx,y)$, $g(x,y) = q^2y^6 g(x,qy)$, which lead us to the following recurrence relations $\mathcal{C}_{m,n} = q^{m-5}\mathcal{C}_{m-6,n}$, $\mathcal{C}_{m,n} = q^{n-4}\mathcal{C}_{m,n-6}$ after some computation. By iteration, we find that, for $0 \le A, B \le 6$, there exists

$$\mathcal{C}_{6m+A,6n+B} = q^{3m^2+3n^2+Am+Bn-2m-n}\mathcal{C}_{A,B}. \tag{1.1}$$

On the other hand, we also find $g(x,y) = -y^2 g(x, 1/y)$, $g(x,y) = -x/y g(y,x)$, which lead us to the following two recurrence relations

$$\mathcal{C}_{m,n} = -\mathcal{C}_{m,-n+2}, \quad \mathcal{C}_{m,n} = -\mathcal{C}_{n+1,m-1}. \tag{1.2}$$

In terms of the relations above, we get

$$\mathcal{C}_{0,0} = -\mathcal{C}_{1,-1} = \mathcal{C}_{1,3} = -\mathcal{C}_{4,0} = \mathcal{C}_{4,2} = -\mathcal{C}_{3,3} = \mathcal{C}_{3,-1} = -\mathcal{C}_{0,2} = \mathcal{C}_{0,0}. \tag{1.3}$$

This shows that 8 of the 36 coefficients sought by (1.3) are equal to $\pm\mathcal{C}_{0,0}$. We can also show that the remaining 28 coefficients are equal to 0 by means of (1.1) and (1.2). For example:

$$\mathcal{C}_{0,1} = -\mathcal{C}_{2,-1} = \mathcal{C}_{2,3} = -\mathcal{C}_{4,1} = \mathcal{C}_{4,1} = 0;\ \mathcal{C}_{0,3} = -\mathcal{C}_{4,-1} = \mathcal{C}_{4,3} = -\mathcal{C}_{4,3} = 0;$$
$$\mathcal{C}_{0,4} = \mathcal{C}_{5,3} = -\mathcal{C}_{4,4} = \mathcal{C}_{4,4} = 0;\ \mathcal{C}_{0,5} = \mathcal{C}_{6,3} = -\mathcal{C}_{4,5} = q\mathcal{C}_{4,3} = 0;\ \cdots$$

Then $g(x,y)$ can be reformulated as follows

$$\begin{aligned} g(x,y) = \mathcal{C}_{0,0}\Bigg\{ &\sum_{m=-\infty}^{\infty} q^{3m^2-2m}(1-x^4q^{4m})x^{6m} \sum_{n=-\infty}^{\infty} q^{3n^2-n}(1-y^2q^{2n})y^{6n} \\ &- \sum_{m=-\infty}^{\infty} q^{3m^2-m}(1-x^2q^{2m})x^{6m+1} \sum_{n=-\infty}^{\infty} q^{3n^2-2n}(1-y^4q^{4n})y^{6n-1}\Bigg\}. \end{aligned}$$

Applying the quintuple product identity, we simplify the last result as

$$g(x,y)=\mathcal{C}_{0,0}$$

$$\times\Big\{\big[q^2,qx^2,q/x^2;q^2\big]_\infty\big[x^4,q^4/x^4;q^4\big]_\infty\big[q^2,y^2,q^2/y^2;q^2\big]_\infty\big[q^2y^4,q^2/y^4;q^4\big]_\infty$$
$$-\frac{x}{y}\big[q^2,x^2,q^2/x^2;q^2\big]_\infty\big[q^2x^4,q^2/x^4;q^4\big]_\infty\big[q^2,qy^2,q/y^2;q^2\big]_\infty\big[y^4,q^4/y^4;q^4\big]_\infty\Big\}.$$

Next, we need to calculate $\mathcal{C}_{0,0}$. Setting $x=q^{1/4}$, $y=i$, we derive $\mathcal{C}_{0,0}=1$ easily. We derive the very identity appeared in the theorem. □

2. Ramanujan's partition congruence $p(11n+6)\equiv 0 \pmod{11}$

As pointed out in [1, 2], applying the quintuple product identity, we redisplayed $f(x,y)$ in Theorem 1 as

$$\frac{f(x,y)}{x^2y^2}=\Big\{\sum_{m=-\infty}^{+\infty}(x^{2+6m}-x^{-2-6m})q^{3m^2+2m}\Big\}\Big\{\sum_{n=-\infty}^{+\infty}(y^{1+6n}-y^{-1-6n})q^{3n^2+n}\Big\}.$$

The difference equation stated in Theorem 1 can be reformulated as

$$\frac{f(x,y)-f(y,x)}{x^2y^2}=\frac{(q;q)_\infty^2}{x^2y}\big[x^2,y^2,xy,x/y,q/x^2,q/y^2,q/xy,qy/x;q\big]_\infty$$
$$=\Big\{\sum_{m=-\infty}^{+\infty}(x^{2+6m}-x^{-2-6m})q^{3m^2+2m}\Big\}\Big\{\sum_{n=-\infty}^{+\infty}(y^{1+6n}-y^{-1-6n})q^{3n^2+n}\Big\}$$
$$-\Big\{\sum_{m=-\infty}^{+\infty}(y^{2+6m}-y^{-2-6m})q^{3m^2+2m}\Big\}\Big\{\sum_{n=-\infty}^{+\infty}(x^{1+6n}-x^{-1-6n})q^{3n^2+n}\Big\}.$$

Applying first $\frac{x\partial}{\partial x}$, and then three times of $\frac{y\partial}{\partial y}$ at $x=y=1$ to both sides, we derive the following identity [1, 2].

Corollary 2.

$$6(q;q)_\infty^{10}=\Big\{\sum_{m=-\infty}^{+\infty}(2+6m)^3q^{3m^2+m}\Big\}\times\Big\{\sum_{n=-\infty}^{+\infty}(1+6n)q^{3n^2+n}\Big\}$$
$$-\Big\{\sum_{m=-\infty}^{+\infty}(2+6m)q^{3m^2+2m}\Big\}\times\Big\{\sum_{n=-\infty}^{+\infty}(1+6n)^3q^{3n^2+n}\Big\}.$$

This identity can be used to prove the Ramanujan's partition congruence $p(11n+6)\equiv 0 \pmod{11}$. Here, using Hirschhorn's method in [5] and Corollary 2, we give another proof of $p(11n+6)\equiv 0 \pmod{11}$.

Theorem 3. *Let $p(n)$ denote the number of unrestricted partitions of the natural number n. Then there holds the congruence*

$$p(11n+6) \equiv 0 \pmod{11}.$$

Proof. By means of Theorem 2, we have

$$\begin{aligned}(q;q)_\infty^{10} \;=\; & 2\Big\{\sum_{m=-\infty}^{+\infty}(2+6m)^3q^{3m^2+m}\Big\}\times\Big\{\sum_{n=-\infty}^{+\infty}(1+6n)q^{3n^2+n}\Big\}\\ -\; & 2\Big\{\sum_{m=-\infty}^{+\infty}(2+6m)q^{3m^2+2m}\Big\}\times\Big\{\sum_{n=-\infty}^{+\infty}(1+6n)^3q^{3n^2+n}\Big\} \pmod{11}.\end{aligned}$$

According to the residues of m, n modulo 11, we may reformulate the summations inside $\{...\}$ as, respectively

$$\begin{aligned}\sum_{n=-\infty}^{+\infty}(1\;+\;6n)q^{3n^2+n} &= I_1-5q^2I_5-4q^4I_7+2q^{14}I_{13}+5q^{24}I_{17}\\ &-\;3q^{30}I_{19}-q^{44}I_{23}+3q^{52}I_{25}+4q^{70}I_{29}-2q^{80}I_{31} \pmod{11};\\ \sum_{n=-\infty}^{+\infty}(1\;+\;6n)^3q^{3n^2+n} &= I_1-4q^2I_5-2q^4I_7-3q^{14}I_{13}+4q^{24}I_{17}\\ &-\;5q^{30}I_{19}-q^{44}I_{23}+5q^{52}I_{25}-2q^{70}I_{29}+3q^{80}I_{31} \pmod{11};\\ \sum_{m=-\infty}^{+\infty}(2\;+\;6m)q^{3m^2+2m} &= 2I_2-4qI_4-3q^5I_8-q^8I_{10}+3q^{16}I_{14}\\ &-\;5q^{21}I_{16}-2q^{33}I_{20}+4q^{56}I_{26}+5q^{65}I_{28}-q^{85}I_{32} \pmod{11};\\ \sum_{m=-\infty}^{+\infty}(2\;+\;6m)^3q^{3m^2+m} &= -3I_2+2qI_4-5q^5I_8+q^8I_{10}+5q^{16}I_{14}\\ &-\;4q^{21}I_{16}+3q^{33}I_{20}-2q^{56}I_{26}+5q^{65}I_{28}-q^{85}I_{32} \pmod{11};\end{aligned}$$

where $I_k = (q^{363+11k}, q^{363-11k}, q^{726}; q^{726})_\infty$. Then

$$\begin{aligned}
&(q;q)_\infty^{10} \pmod{11}\\
=\ & (I_1I_2-q^{22}I_{10}I_{13}-q^{33}I_1I_{20}-q^{44}I_2I_{23}+q^{77}I_{20}I_{23}+q^{88}I_{10}I_{31}+q^{99}I_{13}I_{32}-q^{165}I_{31}I_{32})\\
+\ & q(I_1I_4-q^{11}I_7I_{10}-q^{44}I_4I_{23}-q^{55}I_1I_{26}+q^{77}I_{10}I_{29}+q^{88}I_7I_{32}+q^{99}I_{23}I_{26}-q^{154}I_{29}I_{32})\\
+\ & 2q^2(I_2I_5-q^{22}I_2I_{17}-q^{33}I_5I_{20}-q^{33}I_{13}I_{16}+q^{55}I_{17}I_{20}+q^{77}I_{13}I_{28}+q^{99}I_{16}I_{31}-q^{143}I_{28}I_{31})\\
+\ & 3q^3(I_4I_5-q^{22}I_4I_{17}-q^{22}I_7I_{16}-q^{55}I_5I_{26}+q^{66}I_7I_{28}+q^{77}I_{17}I_{26}+q^{88}I_{16}I_{29}-q^{132}I_{28}I_{29})\\
+\ & 5q^4(I_2I_7-q^{11}I_4I_{13}-q^{33}I_7I_{20}-q^{66}I_2I_{29}+q^{66}I_{13}I_{26}+q^{77}I_4I_{31}+q^{99}I_{20}I_{29}-q^{132}I_{26}I_{31})\\
-\ & 4q^5(I_1I_8-q^{11}I_1I_{14}-q^{33}I_{10}I_{19}-q^{44}I_8I_{23}+q^{55}I_{10}I_{25}+q^{55}I_{14}I_{23}+q^{110}I_{19}I_{32}-q^{132}I_{25}I_{32})\\
+\ & 4q^7(I_5I_8-q^{11}I_5I_{14}-q^{22}I_8I_{17}-q^{33}I_{14}I_{17}-q^{44}I_{16}I_{19}+q^{66}I_{16}I_{25}+q^{88}I_{19}I_{28}-q^{110}I_{25}I_{28})\\
-\ & 5q^{19}(I_8I_{13}-q^{11}I_2I_{19}+q^{11}I_{13}I_{14}+q^{33}I_2I_{25}+q^{44}I_{19}I_{20}-q^{66}I_8I_{31}+q^{66}I_{20}I_{25}+q^{77}I_{14}I_{31})\\
-\ & 3q^9(I_7I_8-q^{11}I_7I_{14}-q^{22}I_4I_{19}-q^{44}I_4I_{25}+q^{66}I_8I_{29}+q^{77}I_{14}I_{29}+q^{77}I_{19}I_{26}-q^{99}I_{25}I_{26})\\
-\ & 2q^{10}(I_5I_{10}-q^{11}I_1I_{16}-q^{22}I_{10}I_{17}+q^{55}I_1I_{28}+q^{55}I_{16}I_{23}-q^{77}I_5I_{32}+q^{99}I_{17}I_{32}-q^{99}I_{23}I_{28}).
\end{aligned}$$

Next, we calculate the summations in $(\cdots)$, respectively. For short, we denote them as $P_i, i=0,1,\cdots,10$. Then the first term can be displayed as

$$P_0 := (I_1-q^{44}I_{23})(I_2-q^{33}I_{20})-q^{22}(I_{10}-q^{66}I_{32})(I_{13}-q^{66}I_{31}).$$

On the other hand, by means of the quintuple product identity, $f(x,y)$ in Theorem 1 can be reformulated as

$$\begin{aligned}
f(x,y) \ =\ & \Big\{y\left[q^6,-qx^6,-q^5/x^6;q^6\right]_\infty - x^4y\left[q^6,-q/x^6,-q^5x^6;q^6\right]_\infty\Big\}\\
\times\ & \Big\{\left[q^6,-q^2y^6,-q^4/x^6;q^6\right]_\infty - y^2\left[q^6,-q^2/x^6,-q^4x^6;q^6\right]_\infty\Big\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&(q;q)_\infty^2\left[x^2,q/x^2,y^2,q/y^2,xy,q/xy,x/y,qy/x;q\right]_\infty\\
=\ & \Big\{\left[q^6,-qx^6,-q^5/x^6;q^6\right]_\infty - x^4\left[q^6,-q/x^6,-q^5x^6;q^6\right]_\infty\Big\}\\
\times\ & \Big\{\left[q^6,-q^2y^6,-q^4/x^6;q^6\right]_\infty - y^2\left[q^6,-q^2/y^6,-q^4y^6;q^6\right]_\infty\Big\}\\
-\ & \Big\{\frac{x}{y}\left[q^6,-qy^6,-q^5/y^6;q^6\right]_\infty - xy^3\left[q^6,-q/y^6,-q^5y^6;q^6\right]_\infty\Big\}\\
\times\ & \Big\{\left[q^6,-q^2y^6,-q^4/y^6;q^6\right]_\infty - x^2\left[q^6,-q^2/x^6,-q^4x^6;q^6\right]_\infty\Big\}.
\end{aligned}$$

Letting $x=q^{44}, y=q^{22}$ in the last identity, we have

$$P_0=\left[q^{22},q^{33},q^{44},q^{55},q^{66},q^{77},q^{88},q^{99},q^{121},q^{121};q^{121}\right]_\infty.$$

We can derive the remaining terms as follows: Letting $x=q^{33}, y=q^{22}$,

$$P_1=\left[q^{11},q^{44},q^{55},q^{55},q^{66},q^{66},q^{77},q^{110},q^{121},q^{121};q^{121}\right]_\infty;$$

Letting $x=q^{44}, y=q^{11}$,

$$P_2=\left[q^{22},q^{33},q^{33},q^{55},q^{66},q^{88},q^{88},q^{99},q^{121},q^{121};q^{121}\right]_\infty;$$

Letting $x = q^{33}, y = q^{11}$,

$$P_3 = \left[q^{22}, q^{22}, q^{44}, q^{55}, q^{66}, q^{77}, q^{99}, q^{99}, q^{121}, q^{121}; q^{121}\right]_\infty;$$

Letting $x = q^{44}, y = q^{33}$,

$$P_4 = \left[q^{11}, q^{33}, q^{44}, q^{55}, q^{66}, q^{77}, q^{88}, q^{110}, q^{121}, q^{121}; q^{121}\right]_\infty;$$

Letting $x = q^{55}, y = q^{22}$,

$$P_5 = \left[q^{11}, q^{33}, q^{44}, q^{44}, q^{77}, q^{77}, q^{88}, q^{110}, q^{121}, q^{121}; q^{121}\right]_\infty;$$

Letting $x = q^{55}, y = q^{11}$,

$$P_7 = \left[q^{11}, q^{22}, q^{44}, q^{55}, q^{66}, q^{77}, q^{99}, q^{110}, q^{121}, q^{121}; q^{121}\right]_\infty;$$

Letting $x = q^{55}, y = q^{44}$,

$$P_8 = \left[q^{11}, q^{11}, q^{22}, q^{33}, q^{88}, q^{99}, q^{110}, q^{110}, q^{121}, q^{121}; q^{121}\right]_\infty;$$

Letting $x = q^{55}, y = q^{33}$,

$$P_9 = \left[q^{11}, q^{22}, q^{33}, q^{55}, q^{66}, q^{88}, q^{99}, q^{110}, q^{121}, q^{121}; q^{121}\right]_\infty;$$

Letting $x = q^{22}, y = q^{11}$,

$$P_{10} = \left[q^{11}, q^{22}, q^{33}, q^{44}, q^{77}, q^{88}, q^{99}, q^{110}, q^{121}, q^{121}; q^{121}\right]_\infty.$$

Applying these results, we derive

$$\begin{aligned}(q;q)_\infty &\equiv P_0 + qP_1 + 2q^2P_2 + 3q^3P_3 + 5q^4P_4 \\ &- 4q^5P_5 + 4q^7P_7 - 5q^{19}P_8 - 3q^9P_9 - 2q^{10}P_{10} \pmod{11}.\end{aligned}$$

By means of the congruence relation

$$\sum_{n=0}^{+\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{(q;q)_\infty^{10}}{(q;q)_\infty^{11}} \equiv \frac{(q;q)_\infty^{10}}{(q^{11};q^{11})_\infty} \pmod{11},$$

we have

$$\begin{aligned}\sum_{n=0}^{+\infty} p(n)q^n \equiv& \frac{P_0}{(q^{11};q^{11})_\infty} + \frac{qP_1}{(q^{11};q^{11})_\infty} + \frac{2q^2P_2}{(q^{11};q^{11})_\infty} + \frac{3q^3P_3}{(q^{11};q^{11})_\infty} + \frac{5q^4P_4}{(q^{11};q^{11})_\infty} \\ &- \frac{4q^5P_5}{(q^{11};q^{11})_\infty} + \frac{4q^7P_7}{(q^{11};q^{11})_\infty} - \frac{5q^{19}P_8}{(q^{11};q^{11})_\infty} - \frac{3q^9P_9}{(q^{11};q^{11})_\infty} - \frac{2q^{10}P_{10}}{(q^{11};q^{11})_\infty} \pmod{11}.\end{aligned}$$

Noting that there are no powers of q congruent to 6 modulo 11, we prove the theorem. □

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