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Solutions for Semipositone Higher-Order Two-Point BVP

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Abstract

In this paper we investigate the following nonlinear semipositone BVP:

$$u^{(4)}(t) = f(t, u(t)), \quad 0 \le t \le 1,\ u'(1) = u''(1) = u'''(1) = 0, \quad ku(0) = u'''(0),$$

where $-6 < k < 0$, $f \ge -M$, and $M \ge 0$. Our approach relies on the Krasnosel'skii fixed point theorem.

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1 Introduction

Recently an increasing interest in studying the fourth-order two-point BVP is observed. Among others we refer to [1, 2, 3, 4, 5]. In this paper we consider the positive solutions of the following BVP:

$$u^{(4)}(t) = f(t, u(t)), \quad 0 \le t \le 1,\ u'(1) = u''(1) = u'''(1) = 0, \quad ku(0) = u'''(0), \tag{1.1}$$

where $-6 < k < 0$, f is continuous and there exists $M > 0$ such that $f \ge -M$. This implies that f is not necessarily nonnegative, monotone, superlinear and sublinear. And also this assumption implies that the problem (1.1) is semipositone .

This paper is organized as follows: in section 2, we present some lemmas. Section 3 is devoted to proving our main results. An example is considered in section 4 to illustrate our main results.

2 Preliminaries and lemmas

Let $C^2[0, 1]$ be the Banach space with norm $\|u\|_0 = \max\{\|u\|, \|u''\|\}$, where

$$\|u\| = \max_{0 \le t \le 1} |u(t)|, \quad u \in C[0, 1].$$

By routine calculation, we easily obtain the following Lemma.

Lemma 2.1. *If $k \neq 0$, then $u^{(4)}(t) = h(t), \quad 0 \le t \le 1,\ u'(1) = u''(1) = u'''(1) = 0, \quad ku(0) = u'''(0)$, has a unique solution $u(t) = \int_0^1 G(t,s)h(s)ds$, where the Green function is*

$$G(t,s) = -\frac{1}{6}\begin{cases} \dfrac{6}{k} + s^3, & 0 \le s \le t \le 1, \\ \dfrac{6}{k} - (s-t)^3 + s^3, & 0 \le t \le s \le 1. \end{cases}$$

Remark 2.2. If $-6 < k < 0$, then

$$0 < (1 + \frac{k}{6})G(0,s) \le G(t,s) \le G(0,s) = \max_{0 \le t \le 1} G(t,s) = -\frac{1}{k} \tag{2.2}$$

in closed bounded region $D = \{(t,s) : 0 \le t \le 1, 0 \le s \le 1\}$.

Let $p(t) := \int_0^1 G(t,s)ds = \frac{1}{24}t^4 - \frac{1}{6}t^3 + \frac{1}{4}t^2 - \frac{1}{6}t - \frac{1}{k}, \quad 0 \le t \le 1.$
Since

$$p'(t) = \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{1}{2}t - \frac{1}{6} = -\frac{1}{6}(1-t)^3 \le 0, \quad 0 \le t \le 1,$$
$$p''(t) = \frac{1}{2}t^2 - t + \frac{1}{2} = \frac{1}{2}(1-t)^2 \ge 0, \quad 0 \le t \le 1,$$

we have

$$\|p\| = \max_{0 \le t \le 1} p(t) = p(0) = -\frac{1}{k}, \quad \min_{0 \le t \le 1} p(t) = p(1) = -\frac{1}{k} - \frac{1}{24}, \tag{2.3}$$
$$\|p''\| = \max_{0 \le t \le 1} |p''(t)| = \frac{1}{2}. \tag{2.4}$$

Lemma 2.3. *Let X be a Banach space, and $K \subset X$ be a cone in X. Assume Ω_1, Ω_2 are bounded open subsets of K with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $F : K \to K$ be a completely continuous operator such that either*

(1) $\|Fu\| \le \|u\|, u \in \partial\Omega_1$, and $\|Fu\| \ge \|u\|, u \in \partial\Omega_2$, or

(2) $\|Fu\| \ge \|u\|, u \in \partial\Omega_1$, and $\|Fu\| \le \|u\|, u \in \partial\Omega_2$.

Then F has a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

To apply the Krasnosel'skii fixed point theorem, we need to construct a suitable cone. Let

$$C_0^2[0,1] = \{u \in C^2[0,1] : u(t) \ge 0,\ u'(1) = u''(1) = u'''(1) = 0,\ ku(0) = u'''(0)\}.$$

Obviously the following set P is a cone in $C^2[0,1]$: $\quad P = \{u \in C_0^2[0,1] : \min_{0 \le t \le 1} u(t) \ge (1+\frac{k}{6})\|u\|\}$, where $-6 < k < 0$. For convenience, let

$$\alpha(r) = \max\{f(t,u) : (t,u) \in D_1(r)\},\ \beta(r) = \min\{f(t,u) : (t,u) \in D_2(r)\}, \tag{2.5}$$

where

$$D_1(r) = \{(t,u) : 0 \le t \le 1,\ \frac{M}{k} \le u \le r + (\frac{1}{k} + \frac{1}{24})M, \},$$

$$D_2(r) = \left\{(t,u) : \frac{1}{4} \le t \le \frac{3}{4},\ (\frac{1}{k} + \frac{175}{6144})M \le u \le r + (\frac{1}{k} + \frac{85}{2048})M, \right\}.$$

$$C_1 = \min\left\{\Big[\max_{0\le t\le 1}\int_0^1 G(t,s)ds\Big]^{-1}, \Big[\max_{0\le t\le 1}\int_0^1 |G''(t,s)|ds\Big]^{-1}\right\} = \min\{-k, 2\},$$

$$C_2 = \max\left\{\Big[\max_{0\le t\le 1}\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)ds\Big]^{-1}, \Big[\max_{0\le t\le 1}\int_{\frac{1}{4}}^{\frac{3}{4}} |G''(t,s)|ds\Big]^{-1}\right\} = \max\left\{\big(-\frac{1}{2k} + \frac{1}{6144}\big)^{-1}, \frac{32}{9}\right\}.$$

Obviously, $0 < C_1 < C_2$.

3 Main results

Theorem 3.1. *Let $-6 < k < 0$. Assume that*

$$f : [0,1] \times [\frac{M}{k}, +\infty) \to [-M, +\infty) \tag{3.6}$$

is continuous, where $M > 0$ is a constant. Suppose there exist two positive numbers r_1 and r_2 with $\min\{r_1, r_2\} > \frac{-6}{6k+k^2}M$ such that

$$\alpha(r_1) \le r_1 C_1 - M, \quad \beta(r_2) \ge r_2 C_2 - M, \tag{3.7}$$

where α, β are as in (2.5), respectively. Then problem (1.1) has at least one positive solution.

Proof. Let $u_0(t) = Mp(t), 0 \le t \le 1$. Then by (2.2) and (2.4) we have

$$(-\frac{1}{k} - \frac{1}{24})M \le u_0(t) \le -\frac{M}{k}, \quad 0 \le t \le 1. \tag{3.8}$$

Consider the fourth-order two-point boundary-value problem

$$u^{(4)}(t) = f(t, u(t) - u_0(t)) + M, \quad 0 \le t \le 1,\ u'(1) = u''(1) = u'''(1) = 0, ku(0) = u'''(0), \tag{3.9}$$

This problem is equivalent to the integral equation $u(t) = \int_0^1 G(t,s)[f(s,u(s) - u_0(s)) + M]ds$.

For $u \in C_0^2[0,1]$, we define the operator A as follows

$$(Au)(t) = \int_0^1 G(t,s)[f(s,u(s) - u_0(s)) + M]ds, \quad 0 \le t \le 1.$$

So, we obtain $(Au)''(t) = \int_t^1 (s-t)[f(s,u(s) - u_0(s)) + M]ds, \quad 0 \le t \le 1$.

Noticing (3.8) and that $u \in C_0^2[0,1]$, we have $\frac{M}{k} \le u(t) - u_0(t) < +\infty, \quad 0 \le t \le 1$.

Thus, from (3.6) we get $(Au)(t) \ge 0, \quad (Au)''(t) \ge 0, \quad t \in [0,1]$. By the definition of $G(t,s)$,

$$G'(1,s) = G''(1,s) = G'''(1,s) = 0, \quad \text{and} \quad G'''(0,s) = kG(0,s) = -1,$$

which implies that $(Au)'(1) = (Au)''(1) = (Au)'''(1) = 0, \quad \text{and} \quad k(Au)(0) = (Au)'''(0)$. Hence, $A : C_0^2[0,1] \to C_0^2[0,1]$. Moreover, for each $t \in [0,1]$, (By (2.2) we have

$$\begin{aligned}(Au)(t) &= \int_0^1 G(t,s)[f(s,u(s)-u_0(s))+M]ds \geq (1+\frac{k}{6})\int_0^1 G(0,s)[f(s,u(s)-u_0(s))+M]ds \\ &\geq (1+\frac{k}{6})\max_{0\leq t\leq 1}\int_0^1 G(t,s)[f(s,u(s)-u_0(s))+M]ds = (1+\frac{k}{6})\|Au\|.\end{aligned}$$

Thus, $A : P \to P$.

We can check that A is completely continuous by routine method. Since $C_1 < C_2$, it is easy to check that $r_1 \neq r_2$. Without loss of generality, we assume $r_1 < r_2$. Let

$$\Omega_1 = \{u \in P : \|u\|_0 < r_1\}, \quad \Omega_2 = \{u \in P : \|u\|_0 < r_2\}.$$

If $u \in \partial\Omega_1$, then $\|u\|_0 = r_1$. So, $\|u\| \leq r_1$. This implies $0 \leq u(t) \leq r_1, \quad 0 \leq t \leq 1$.

By (2.3), for $0 \leq t \leq 1$, we have $\frac{1}{k}M \leq u(t) - u_0(t) \leq r_1 + \left(\frac{1}{k} + \frac{1}{24}\right)M$.

By (3.7), $f(t, u(t) - u_0(t)) \leq \alpha(r_1) \leq r_1C_1 - M, \quad 0 \leq t \leq 1$. It follows that

$$\|Au\| = \max_{0\leq t\leq 1}\int_0^1 G(t,s)[f(s,u(s)-u_0(s))+M]ds \leq r_1C_1\max_{0\leq t\leq 1}\int_0^1 G(t,s)ds \leq r_1,$$

$$\|(Au)''\| = \max_{0\leq t\leq 1}\int_0^1 |G''(t,s)|[f(s,u(s)-u_0(s))+M]ds \leq r_1C_1\max_{0\leq t\leq 1}\int_0^1 |G''(t,s)|ds \leq r_1.$$

Therefore, $\|Au\|_0 \leq r_1 = \|u\|_0$.

If $u \in \partial\Omega_2$, then $\|u\|_0 = r_2$. So, $\|u\| \leq r_2$. This implies that $0 \leq u(t) \leq r_2, \quad 0 \leq t \leq 1$. Since

$$-\frac{85}{2048} - \frac{1}{k} = p(\frac{3}{4}) \leq p(t) \leq p(\frac{1}{4}) = -\frac{175}{6144} - \frac{1}{k}, \quad \frac{1}{4} \leq t \leq \frac{3}{4},$$

we have

$$(\frac{1}{k} + \frac{175}{6144})M \leq u(t) - u_0(t) \leq r_2 + (\frac{1}{k} + \frac{85}{2048})M, \quad \frac{1}{4} \leq t \leq \frac{3}{4},$$

Thus, by (3.7) we obtain $f(t, u(t) - u_0(t)) \geq \beta(r_2) \geq r_2C_2 - M, \quad \frac{1}{4} \leq t \leq \frac{3}{4}$. From this,

$$\|Au\| \geq \max_{0\leq t\leq 1}\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)[f(s,u(s)-u_0(s))+M]ds \geq r_2C_2\max_{0\leq t\leq 1}\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)ds \geq r_2,$$

and $\|(Au)''\| \geq \max_{0\leq t\leq 1}\int_{\frac{1}{4}}^{\frac{3}{4}} G''(t,s)[f(s,u(s)-u_0(s))+M]ds \geq r_2C_2\max_{0\leq t\leq 1}\int_{\frac{1}{4}}^{\frac{3}{4}} G''(t,s)ds \geq r_2$.

It follows that $\|Au\|_0 \geq r_2 = \|u\|_0$. By Lemma 2.3, the operator A has at least a fixed point $\overline{u} \in P$ with $r_1 \leq \|\overline{u}\|_0 \leq r_2$. So (3.9) has at least one solution $\overline{u} \in P$ with $r_1 \leq \|\overline{u}\|_0 \leq r_2$.

Let $u_*(t) = \overline{u}(t) - u_0(t)$, $0 \leq t \leq 1$. Then u_* is a solution of (1.1). In fact, since $A\overline{u} = \overline{u}$, we have

$$u_*(t) + u_0(t) = \overline{u}(t) = (A\overline{u})(t) = \int_0^1 G(t,s)[f(s,\overline{u}(s)-u_0(s))+M]ds = \int_0^1 G(t,s)f(s,u_*(s))ds + u_0(t).$$

It follows that $u_*(t) = \int_0^1 G(t,s)f(s,u_*(s))ds, \quad 0 \le t \le 1$.

In other words, u_* is a solution of (1.1). And $u_* + u_0 \in P$ and $r_1 \le \|u^* + u_0\|_0 \le r_2$.

Since $r_1 = \min\{r_1, r_2\} > -\dfrac{6}{6k+k^2}M$, we have

$$\begin{aligned} u_*(t) &= [u_*(t) + u_0(t)] - u_0(t) = [u_*(t) + u_0(t)] - Mp(t) \\ &\ge (1+\frac{k}{6})\|u_*(t) + u_0(t)\| + \frac{M}{k} \ge (1+\frac{k}{6})[r_1 + \frac{6}{6k+k^2}M] > 0, \quad 0 \le t \le 1, \end{aligned}$$

which implies that u_* is a positive solution of (1.1). □

Using Theorem 3.1, we can prove following result.

Theorem 3.2. *Let $-6 < k < 0$. Assume that*

$$f : [0,1] \times [\frac{M}{k}, +\infty) \to [-M, +\infty) \tag{3.10}$$

is continuous, where $M \ge 0$ is a constant. Suppose that there exist three positive numbers $r_1 < r_2 < r_3$ with $r_1 > -\dfrac{6}{6k+k^2}M$ such that one of the following conditions is satisfied:

(1) $\alpha(r_1) \le r_1C_1 - M$, $\beta(r_2) > r_2C_2 - M$, $\alpha(r_3) \le r_3C_1 - M$;

(2) $\beta(r_1) \ge r_1C_2 - M$, $\alpha(r_2) < r_2C_1 - M$, $\beta(r_3) \ge r_3C_2 - M$.

Then problem (1.1) *has at least two positive solutions.*

4 Examples

Example 4.1. Consider the boundary-value problem

$$u^{(4)}(t) = f(t,u(t)), \quad 0 \le t \le 1, \; u'(1) = u''(1) = u'''(1) = 0, \quad -2u(0) = u'''(0), \tag{4.11}$$

where $f : [0,1] \times [-1, +\infty) \to [-2, +\infty)$ is defined by

$$f(t,u) = \begin{cases} t^2 + \sqrt{u+1} - 2, & (t,u) \in [0,1] \times [-1, -\frac{1}{2}], \\ t^2 + \dfrac{u}{4} + \dfrac{\sqrt{2}}{2} - \dfrac{15}{8}, & (t,v) \in [0,1] \times [-\dfrac{1}{2}, \infty), \\ t^2 + \sqrt{u+1} + \dfrac{9}{2}\sqrt{2} - \dfrac{19}{10}, & (t,u) \in [0,1] \times [-1, -\dfrac{1}{2}], \\ t^2 + \dfrac{u}{4} + 5\sqrt{2} - \dfrac{71}{40}, & (t,u) \in [0,1] \times [-\dfrac{1}{2}, \infty). \end{cases}$$

Thus, $k = -2$, $M = 2$, $C_1 = 2$ and $C_2 = \frac{6144}{1537}$. For

$$D_1(r) = \big\{(t,u) : 0 \le t \le 1, \; -1 \le u \le r - \frac{11}{12}\big\}, D_2(r) = \big\{(t,u) : \frac{1}{4} \le t \le \frac{3}{4}, \; -\frac{2897}{3072} \le u \le r - \frac{939}{1024}\big\}.$$

By simple computations, we obtain

$$\alpha(6) = \max\left\{f(1,\frac{61}{12}),\ f(1,\frac{61}{12}),\ f(1,-\frac{1}{2}),\ f(1,-\frac{1}{2})\right\} = f(1,\frac{61}{12}) = 8.76 < 10 = 6C_1 - M,$$

and

$$\beta(\frac{13}{8}) = \min\left\{f(\frac{1}{4},-\frac{2897}{3072}),\ f(\frac{1}{4},-\frac{2897}{3072}),\ f(\frac{1}{4},-\frac{1}{2}),\ f(\frac{1}{4},-\frac{1}{2}\right\} = f(\frac{1}{4},-\frac{2897}{3072}) = 4.76 > \frac{13}{8}C_2 - M.$$

Take $r_1 = 6$ and $r_2 = \frac{13}{8}$. Then (3.7) holds. Moreover, we have $\min\{r_1, r_2\} = \frac{13}{8} > \frac{3}{2} = -\frac{6}{6k+k^2}M$. So, by Theorem 3.1, problem (4.11) has at least one positive solution.

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