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Common Fixed Point Theorems in Fuzzy Metric Spaces

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Abstract

In this paper we prove some common fixed point theorems for a pair of self mappings which possess the property E.A and satisfy certain sufficient conditions in the setting of a fuzzy metric space.

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1 Introduction

Ever since the concept of fuzzy sets was introduced by Zadeh[10], many authors have expansively developed the theory of fuzzy sets and its applications. Kramosil and Michalek[4], Kaleva and Seikkala[13], George and Veeramani[1] introduced the concept of fuzzy metric spaces in different ways. Kramosil and Michalek[4] and later Grabiec[2] obtained the fuzzy version of Banach contraction principle. Many authors proved fixed point theorems for contractive maps in fuzzy metric spaces. In 1986 Jungck[3] generalized the concept of commutativity by introducing compatibility. Mishra et al[9] proved common fixed point theorems for compatible maps on fuzzy metric spaces. Sushil Sharma[11] obtained common fixed point theorems for six self mappings satisfying compatibility of type α conditions. Aamir and El Moutawakil[8] further generalized the concept of non compatibility by introducing the notion of E.A property in the classical settings of a metric space. In this paper we extend the concept of E.A property to fuzzy metric space and obtain common fixed point theorems for a pair of self mappings under sufficient contractive type conditions. Now we begin with some known definitions and preliminary concepts

Preliminaries

Definition 1.1. *A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is called a continuous t norm if* $([0,1], *)$ *is an abelian toplogical monoid with unit* 1 *such that* $a*b \leq c*d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Examples of t**- norm are** $a * b = ab$ **and** $a * b = min\{a, b\}$

Definition 1.2. *The* 3*-tuple*$(X, M, *)$ *is called a fuzzy metric space, if* X *is an arbitrary set,*$*$ *is a continuous* t*-norm and* M *is a fuzzy set in* $X^2 \times [0, \infty]$ *satisfying the following conditions:*
For all x, y, z *in* X *and* $s, t > 0$

(i) $M(x, y, 0) = 0$

(ii) $M(x, y, t) = 1$*, for all* $t > 0$*,if and only if* $x = y$

(iii) $M(x, y, t) = M(y, x, t)$

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(v) $M(x, y, .) : [0, \infty) \to [0, 1]$ *is left continuous,*

(vi) $\lim_{t\to\infty} M(x, y, t) = 1$.

Definition 1.3. *Let* $(X, M, *)$ *be a fuzzy metric space*
A sequence $\{x_n\}$ *in* X *is called Cauchy sequence if and only if* $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ *for each* $p > 0$*,*$t > 0$.
A sequence $\{x_n\}$ *in* X *is said to converge to* x *in* X *if and only if* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *for each* $t > 0$.

A fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent in X.

Lemma 1.4. *[2] $M(x, y, .)$ is nondecreasing for all x, y in X.*

Definition 1.5. *Two self mappings S and T of a fuzzy metric space $(X, M, *)$ are said to be weakly commuting if $M(STx, TSx, t) \geq M(Sx, Tx, t)$ for all $t > 0$ and for all $x \in X$*

Clearly two commuting mappings are weakly commuting

Definition 1.6. *Let T and S be two self mappings of a fuzzy metric space $(X, M, *)$. S and T are said to be compatible if $\lim_{n\to\infty} M(STx_n, TSx_n, t)) = 1$ whenever (x_n) is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x_0$*

Obviously two weakly commuting mappings are compatible.

Definition 1.7. *Two self mappings T and S of a fuzzy metric space $(X, M, *)$ are said to be weakly compatible if they commute at their coincidence points;(i.e) if $Tu = Su$ for some $u \in X$, then $TSu = STu$.*

It is easy to see that two compatible maps are weakly compatible.

2 Main Results

Definition 2.1. *Let $S$and T be two self mappings of a fuzzy metric space $(X, M, *)$. We say that T and S satisfy E.A property , if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x_0$, for some $x_0 \in X$; i.e $\lim_{n\to\infty} M(Tx_n, x_0, t) = \lim_{n\to\infty} M(Sx_n, x_0, t) = 1$ for all $t \in [0, \infty)$*

Example 2.2. *Let $X = [0, \infty)$.Let $M(x, y, t) = \frac{t}{t+|x-y|}$.*
Define $T, S : X \to [0, \infty)$ by $Tx = \frac{x}{5}$and $Sx = \frac{2x}{5}$ for all $x \in X$.Then $lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0$, where $x_n = \frac{1}{n}$

Lemma 2.3. *[9] If for all $x, y \in X$, $t > 0$ with positive number $q \in (0, 1)$ and $M(x, y, qt) \geq M(x, y, t)$, then $x = y$.*

Theorem 2.4. *Let S and T be two weakly compatible self mappings of a fuzzy metric space $(X, M, *)$ with $t * t \geq t$ such that* *(i) T and S satisfy the E.A property*

(ii) For every $x \neq y \in X$ and for $t > 0$

$$M(Tx, Ty, qt) \geq \min\{M(Sx, Sy, t), [M(Tx, Sx, t) * M(Ty, Sy, t)], [M(Ty, Sx, t) * M(Tx, Sy, t)]\},$$

where $0 < q < 1$

(iii) $T(X) \subset S(X)$

(iv) $S(X)$ or $T(X)$ is complete subspace of X

Then T and S have a unique common fixed point.

Proof. Since T and$S$ satisfy the E.A property, there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty}Tx_n = lim_{n\to\infty}Sx_n = x_0$ for some $x_0\ inX$. Suppose that $S(X)$ is complete.Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$.
$\therefore lim_{n\to\infty}Tx_n = Sa$ by (i).
Now we show that $Ta = Sa$.
Condition (ii) implies that

$$M(Tx_n, Ta, qt) \geq \min\{M(Sx_n, Sa, t), [M(Tx_n, Sx_n, t) * M(Ta, Sa, t)], [M(Ta, Sx_n, t) * M(Tx_n, Sa, t)]\}$$

Letting limit $n \to \infty$

$$\begin{aligned} M(Sa, Ta, qt) &\geq \min\{1, [M(Sa, Sa, t) * M(Ta, Sa, t)], [M(Ta, Sa, t) * 1]\} \\ &= \min\{1, M(Ta, Sa, t), M(Ta, Sa, t)\} \\ M(Sa, Ta, qt) &\geq M(Ta, Sa, t) \\ \therefore Ta = Sa & \end{aligned}$$

Now we show that Ta is the common fixed point of T and S. Since T and S are weakly compatible, $STa = TSa = SSa = TTa$.

$$\begin{aligned} M(Ta, TTa, qt) \geq& \\ & \min\{M(Sa, STa, t), [M(Ta, Sa, t) * M(TTa, STa, t)], [M(TTa, Sa, t) * M(Ta, STa, t)]\} \\ =& \min\{M(Ta, TTa, t), M(Ta, Ta, t) * M(TTa, TTa, t)], [M(TTa, Ta, t) * M(Ta, TTa, t)]\} \\ =& \min\{M(Ta, TTa, t), 1, M(Ta, TTa, t) * M(Ta, TTa, t)\} \end{aligned}$$

$M(Ta, TTa, qt) \geq M(Ta, TTa, t)$
$\therefore TTa = Ta$
Hence Ta is the common fixed point of T and S
Even, if we assume that $T(X)$ is complete and proceed as above the result will be same.
Now it is left to prove that the fixed point is unique.
Let x_0 and y_0 be two common fixed points of T and S.
Then

$$\begin{aligned} M(x_0, y_0, qt) =& M(Tx_0, Ty_0, qt) \\ \geq& \; min\{M(Sx_0, Sy_0, t), [M(Tx_0, Sx_0, t) * M(Ty_0, Sy_0, t)], [M(Ty_0, Sx_0, t) * M(Tx_0, Sy_0, t)]\} \\ =& \min\{M(x_0, y_0, t), [M(x_0, x_0, t) * M(y_0, y_0, t)], [M(y_0, x_0, t) * M(x_0, y_0, t)]\} \\ =& \min\{M(x_0, y_0, t), 1, [M(y_0, x_0, t) * M(x_0, y_0, t)]\} \end{aligned}$$

$M(x_0, y_0, qt) \geq M(x_o, y_0, t)$
$\therefore x_0 = y_0$. □

Corollary 2.5. *Let S and T be two noncompatible weakly compatible self mappings of a fuzzy metric space $(X, M, *)$ with $t * t \geq t$such that*

$$(i) M(Tx, Ty, qt) \geq \\ min\{M(Sx, Sy, t), [M(Tx, Sx, t) * M(Ty, Sy, t)], [M(Ty, Sx, t) * M(Tx, Sy, t)]\}$$

(ii)$T(X) \subset S(X)$.
If $S(X)$ or $T(X)$ is complete subspace of X, then T and S have a unique common fixed point

Theorem 2.6. *Let S and T be two weakly compatible self mappings of a fuzzy metric space $(X, M, *)$ with $t * t \geq t$ such that* *(i) T and S satisfy the E.A property*

(ii) For every $x \neq y \in X$ and for $t > 0$

$$M(Tx, Ty, qt) \geq \\ min\{M(Sx, Sy, t), \frac{[M(Tx, Sx, t) + M(Ty, Sy, t)]}{2}, \frac{[M(Tx, Sy, t) + M(Ty, Sx, t)]}{2}\}$$

(iii) $T(X) \subset S(X)$

(iv) $S(X)$ or $T(X)$ is complete subspace of X

Then T and S have a unique common fixed point.

Proof. Since T and$S$ satisfy the E.A property, there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty}Tx_n = lim_{n\to\infty}Sx_n = x_0$ for some $x_0\ inX$. Assuming $S(X)$ to be complete,we get $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$.
$\therefore lim_{n\to\infty}Tx_n = Sa$ by (i).
We claim that $Ta = Sa$.
If $Ta \neq Sa$, then $M(Ta, Sa, t) < 1$ for all t.
Condition (ii) implies that

$$M(Tx_n, Ta, qt) \geq \\ min\{M(Sx_n, Sa, t), \frac{[M(Tx_n, Sx_n, t) + M(Ta, Sa, t)]}{2}, \frac{[M(Tx_n, Sa, t) + M(Ta, Sx_n, t]}{2}\}$$

Letting limit $n \to \infty$

$$\begin{aligned} M(Sa, Ta, qt) &\geq min\{1, \frac{[1 + M(Ta, Sa, t)]}{2}, \frac{[1 + M(Ta, Sa, t]}{2}\} \\ M(Sa, Ta, qt) &\geq \frac{[1 + M(Ta, Sa, t)]}{2} \\ &\geq M(Ta, Sa, t) \end{aligned}$$

for all t
Because,if $\frac{[1+M(Ta,Sa,t)]}{2} \leq M(Ta, Sa, t)$, then $M(Ta, Sa, t) > 1$,which is a contradiction.
Hence Ta = Sa.

Let us prove now that Ta is the common fixed point of T and S.
Suppose that $Ta \neq TTa$.
Since T and S are weakly compatible $STa = TSa = SSa = TTa$.

$$
\begin{aligned}
&M(Ta, TTa, qt) \geq \\
&\qquad min\{M(Sa, STa, t), \frac{[M(Ta, Sa, t) + M(TTa, STa, t)]}{2}, \frac{[M(Ta, STa, t) + M(TTa, Sa, t]}{2}\} \\
&= \; min\{M(Ta, TTa, t), \frac{[1 + M(TTa, TTa, t)]}{2}, \frac{[M(Ta, TTa, t) + M(TTa, Ta, t]}{2}\} \\
&= \; min\{M(Ta, TTa, t), \frac{[1 + M(TTa, TTa, t)]}{2}, M(Ta, TTa, t)\} \\
&= \; M(Ta, TTa, t)
\end{aligned}
$$

Because, $M(Ta, TTa, t) \leq \frac{[1+M(TTa,TTa,t)]}{2}$
Thus $M(Ta, TTa, qt) \geq M(Ta, TTa, t)$ for all t.
This implies $TTa = Ta$
Hence Ta is the common fixed point of T and S
As in the above theorem 2.4,the proof will be similar if we assume that $T(X)$ is complete subspace of X Finally we show that the fixed point is unique.
If possible, let x_0 and y_0 be two common fixed points of T and S Then

$$
\begin{aligned}
&M(x_0, y_0, qt) = M(Tx_0, Ty_0, qt) \\
&\geq \; min\{M(Sx_0, Sy_0, t), \frac{[M(Tx_0, Sx_0, t) + M(Ty_0, Sy_0, t)]}{2}, \frac{[M(Tx_0, Sy_0, t) + M(Ty_0, Sx_0, t)]}{2}\} \\
&= \; min\{M(x_0, y_0, t), \frac{[M(x_0, x_0, t) + M(y_0, y_0, t)]}{2}, \frac{[M(x_0, y_0, t) + M(y_0, x_0, t)]}{2}\} \\
&= \; min\{M(x_0, y_0, t), 1, M(x_0, y_0, t)\}
\end{aligned}
$$

$\Rightarrow M(x_0, y_0, qt) \geq M(x_o, y_0, t)$, for all t.
$\therefore x_0 = y_0$.
Hence the theorem.

Corollary 2.7. *Let S and T be two noncompatible weakly compatible self mappings of a fuzzy metric space $(X, M, *)$ such that*

$$
\begin{aligned}
&(i) M(Tx, Ty, qt) \geq \\
&\qquad min\{M(Sx, Sy, t), \frac{[M(Tx, Sx, t) + M(Ty, Sy, t)]}{2}, \frac{[M(Tx, Sy, t) + M(Ty, Sx, t)]}{2}\}
\end{aligned}
$$

(ii)$T(X) \subset S(X)$.
If $S(X)$ *or* $T(X)$ *is complete subspace of* X*, then* T *and* S *have a unique common fixed point.*

□

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