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# Characterization of Isometric Operators through Remotality

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**Abstract**

There are many characterizations of isometric operators between Banach spaces. In this paper we give a new characterization of isometries through proximinal sets and remotal sets.

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## I.Introduction

Let $X$ be a Banach space and $L(X, X)$ be the space of all bounded linear operators from $X$ to $X$. An operator $T \in L(X, X)$ is called an **isometry** if $\|Tx\| = \|x\|$ for all $x \in X$.Characterization of isometries in Banach spaces is an important problem in the geometric theory of Banach spaces. So much work has been done on determining the isometries of certain function and operator spaces. We refer to [1 ], [2 ] and [3 ] and the references there in for more on the form of isometries on different Banach spaces. In [ 4], isometries were characterized using proximinal sets in Banach spaces, where proof uses orthogonality in Banach spaces. In this paper we give a very simple proof of such a result and give a new characterization of isometric operators using remotal sets in Banach spaces. Throughout this paper, we let $B[X] = \{x \in$

$X : \|x\| \leq 1\}$, and $S_k(X) = \{x \in X : \|x\| = k\}$. Isometries considered in this paper are onto isometries.

## II. Isometries Via Remotal Sets

Let $X$ be a Banach space and $E$ be a closed subset of $X$. For $x \in X$, we let

$$d(x, E) = \inf\{\|x - e\| : e \in E\},\ P(x, E) = \{y \in E : d(x, E) = \|x - y\|\}$$

$$D(x, E) = \sup\{\|x - e\| : e \in E\},\ \ F(x, E) = \{y \in E : D(x, E) = \|x - y\|\}$$

$E$ is called **proximinal** if $P(x, E)$ is not empty, and is called **remotal** if $F(x, E)$ is not empty. We refer to [ ] and [ ] for results on remotal sets, and we refer to [ ] for results on proximinal sets.

For $T \in L(X, X)$, we write $T(E) = \{Ty : y \in E\}$.

**Definition 2.1.** An operator $T \in L(X, X)$ is said to **preserve remotality** (proximinality [ ]) if $F(Tx, T(E)) = T(F(x, E))$ $(P(Tx, T(E) = T(P(x, E))$. The operator $T$ is called **k−isometric** operator for some $k \in (0, \infty)$ if $T = kU$ for some isometric operator $U$.

One can easily see that every isometry on a Banach space preserves remotality and proximinality. The main result of this section is the following.

**Theorem 2.2**. Let $T \in L(X, X)$. The followings are equivalent

$(i)$ $T$ is $k-$ isometry

$(ii)T$ preserves remotality

$(iii)$ $T$ preserves proximinality

**Proof.** $(i) \rightarrow (ii)$. Let $T$ be a $k-$ isometric operator on the Banach space $X$.Then $T = kU$, where $U$ is an isometric operator. Let $E$ be any remotal set in $\dot{X}$ and $x \in X$. Let $y \in F(Tx, T(E))$.Then $\|Tx - y\| \geq \|Tx - z\|$ for all $z \in T(E)$. Since $y, z \in T(E)$, then $y = Tx_1$ and $z = Tx_2$. Hence $\|Tx - Tx_1\| \geq \|Tx - Tx_2\|$ . Consequently, $k\|x - x_1\| \geq k\|x - x_2\|$, and so $\|x - x_1\| \geq \|x - x_2\|$ .Since this is true for all $x_2 \in E$, it follows that $x_1 \in F(x, E)$, and $Tx_1 \in F(Tx, T(E))$. So preserves remotality.

The proof of $(i) \rightarrow (iii)$ follows the same steps as in $(i) \rightarrow (ii)$, where $\leq$ will replace $\geq$ .

Now, for $(ii) \rightarrow (i)$.First, we remark that an operator $T \in L(X, X)$ is $k-$ isometry if and only if $T(S_1(X)) = S_k(X)$. Indeed, for such $T$, $\|T\| = k$. Further $\frac{T}{k}$ is an isometry. Hence $T = k\ \frac{T}{k}$ as required.

Now, let $E = S_1(X)$, and $x = 0$. Then $F(x, E) = E = S_1(X)$.Since $T$ preserves remotality, then $F(Tx, T(E)) = F(0, T(E)) = T(F(x, E)) = T(S_1(X))$. Now, we have to notice that for $z, w \in F(0, T(E))$ one has $\|z\| = \|w\|$ .Hence $\|Ta\| = \|Tb\| = k$, say, for all $a, b \in S_1(X)$. Thus $T(S_1(X)) = S_k(X)$, and consequently, $T$ is a multiple of an isometry.

$(iii) \to (i)$ Again, the proof follows by taking $E = S_1(X)$, and $x = 0$.Then $P(x, E) = E = S_1(X)$, and the rest of the proof follows the same lines as $(ii) \to (i)$.

We have to remark that $(iii) \to (i)$ was proved in [ ] using orthogonality in Banach spaces. Our proof is much more simpler and appealing.

A nice consequence of Theorem 2.2 is:

**Corollary 2.3**. An onto operator in $L(X, X)$ preserves remotality if and only if it preserves proximinality.

## III. Invariant Points Under Isometries.

Let $X$ be a Banach space. An element $x \in S_1[X]$ is called a smooth point if there is a unique linear functional $x^* \in S_1[X]$ such that $< x^*, x >= \|x\| = 1$. An element $x \in S_1[X]$ is called an exposed point if there exists a linear functional $x^* \in X^*$ such that $< x^*, x >= \|x\| = 1$, but $(< x^*, y >) < 1$ for all $y \neq x$ in $S_1[X]$.

**Definition 3.1**. Let $\alpha > 0$. An element $x \in B[X]$ is called an $\alpha-$extreme point if there is no $y \in X$, with $\|y\| = \alpha$, such that $\|x \pm y\| \leq 1$.

One can easily see that $x \in B[X]$ is an extreme point if it is $\alpha-$extreme for all $\alpha \in [0, 1]$. An example of $\alpha-$extreme points is the following.

Let $X = (R^2, \|\|_\infty)$. Take $x = (\frac{9}{10}, 1)$. Then $x$ is $\alpha-$extreme for all $\alpha \in (\frac{1}{10}, 1]$.

Now we state

**Theorem 3.2**. Let $X$ be a Banach space and $T \in L(X)$ be an isometric onto operator. Then:

$(i)$ $Tx$ is smooth if and only if $x$ is smooth.

$(ii)$ $Tx$ is exposed if and only if $x$ is exposed

$(iii)$ $Tx$ is $\alpha-$extreme if and only if $x$ is $\alpha-$extreme

**Proof**. $(i)$ and $(ii)$ are known in the literature and easy to prove.

As for $(iii)$, let $x \in B[X]$ be $\alpha-$extreme for some $\alpha > 0$. If $Tx$ is not $\alpha-$extreme then there is some $y \in X$ such that $\|y\| = \alpha$, and

$\|Tx \pm y\| \leq 1$. Since $T$ is an isometric onto operator, then $y = Tz$ for some $z \in X$ with $\|z\| = \alpha$.But then $\|Tx \pm Tz\| \leq 1$. Once again

since $T$ is an isometry, we get $\|x \pm z\| \leq 1$, and $x$ is not $\alpha-$ extreme, which is a contradiction.

The converse follows the same line of proof and will be omitted.

**IV. Further Results**

In this section, we present a result on the isometries of p-summing operators.

**Lemma 4.1**. Let $(x_n) \in \ell^p < X >$ and $A \in L(X)$ be an isometric onto operator. Then $(Ax_n) \in \ell^p < X >$ . Moreover, $\|(Ax_n)\|_{\varepsilon(p)} = \|(x_n)\|_{\varepsilon(p)}$ .

**Proof.** Let $A \in L(X)$ be an isometric onto operator and $(x_n) \in \ell^p < X >$ . We claim that $(Ax_n) \in \ell^p < X >$ . Indeed,

$$\begin{aligned} \sup_{\|x^*\|=1} \sum_{n=1}^{\infty} |< Ax_n, x^* >|^p &= \sup_{\|x^*\|=1} \sum_{n=1}^{\infty} |< x_n, A^*x^* >|^p \\ &= \sup_{\|z^*\|=1} \sum_{n=1}^{\infty} |< x_n, z^* >|^p, \quad [z^* = A^*x^*] \end{aligned}$$

noting that $A^* (S_1 [X^*]) = S_1 [X^*]$ since $A^*$ is an isometric onto operator. Thus $(Ax_n) \in \ell^p < X >$ and $\|(Ax_n)\|_{\varepsilon(p)} = \|(x_n)\|_{\varepsilon(p)}$ . ■

**Theorem 4.2.** Let $J : \pi_p(X) \to \pi_p(X)$ be a bounded linear operator. If $J(T) = TA$(or $AT$) $\forall T \in \pi_p(X)$, where $A \in L(X)$ is an isometric onto operator, then $J$ is isometric onto.

**Proof.** Let $J : \pi_p(X) \to \pi_p(X)$ be bounded and linear such that $J(T) = TA \ \forall T \in \pi_p(X)$, where $A$ is an isometric onto operator on $X$. Since $\pi_p(X)$ is an operator ideal it follows that $J(T) = TA \in \pi_p(X)$. Thus by Theorem 4.1.6, we have

$$\begin{aligned} \|J(T)\|_{\pi(p)} &= \|TA\|_{\pi(p)} \\ &= \|F_{TA}\|_{\pi(p)}, \qquad [F \in L(\ell^p < X >, \ell^p(X))] \\ &= \sup_{\|x_n\|_{\varepsilon(p)}=1} \|F_{TA}(x_n)\|_{\alpha(p)} \\ &= \sup_{\|x_n\|_{\varepsilon(p)}=1} \left( \sum_{n=1}^{\infty} \|TA(x_n)\|^p \right)^{\frac{1}{p}} \\ &= \sup_{\|x_n\|_{\varepsilon(p)}=1} \left( \sum_{n=1}^{\infty} \|Ty_n\|^p \right)^{\frac{1}{p}}, \quad [(y_n) = (Ax_n)] \end{aligned}$$

But by Lemma 4.2.1, $\|(y_n)\|_{\varepsilon(p)} = \|(x_n)\|_{\varepsilon(p)}$. Hence,

$$\begin{aligned}\|J(T)\|_{\pi(p)} &= \sup_{\|x_n\|_{\varepsilon(p)}=1} \left( \sum_{n=1}^{\infty} \|Ty_n\|^p \right)^{\frac{1}{p}} \\ &= \sup_{\|y_n\|_{\varepsilon(p)}=1} \left( \sum_{n=1}^{\infty} \|Ty_n\|^p \right)^{\frac{1}{p}} \\ &= \|F_T\| \\ &= \|T\|_{\pi(p)}.\end{aligned}$$

In order to show that $J$ is onto, let $S \in \pi_p(X)$. Since $A$ is an isometric onto operator, then $A^{-1}$ is an isometric onto operator. Consequently, $SA^{-1} \in \pi_p(X)$. Further, $J(SA^{-1}) = (SA^{-1})\ \ A = S$.

Similarly, if $J(T) = AT$, then we can easily show that $\|T\,(T)\|_{\pi(p)} = \|T\|_{\pi(p)}$. Further, if $S \in \pi_p(X)$ then $T = A^{-1}S \in \pi_p(X)$ and $J(T) = S$. Hence, $J$ is an isometric onto operator. ∎

**Theorem 4.4**. Let $J : D_1(X) \to D_1(X)$ be a bounded linear operator. If $J(T) = TA$ (or $AT$) $\ \forall T \in D_1(X)$, where $A \in L(X^{**})$ is isometric onto, then $J$ is isometric onto.

**Proof.** Let

$$E = \left\{ (x_n^*) \in \ell^1 < X^* >: \|Tx\| \leq \left( \sum_{i=1}^{\infty} |< x, x_n^* >| \right) \quad \forall x \in X \right\},$$

and

$$F = \left\{ (z_n^*) \in \ell^1 < X^* >: \|TA(x)\| \leq \left( \sum_{i=1}^{\infty} |< x, z_n^* >| \right) \quad \forall x \in X \right\}.$$

We claim that $\forall\,(z_n^*) \in F, \quad (z_n^*) = (A^* x_n^*)$ for some $(x_n^*) \in E$, and if $(x_n^*) \in E$, then $(A^* x_n^*) \in F$. So let $(x_n^*) \in E$. Then

$$\begin{aligned}\|TA\,(x)\| &= \|T\,(Ax)\| \\ &\leq \left( \sum_{n=1}^{\infty} |< Ax, x_n^* >| \right) \\ &\leq \left( \sum_{n=1}^{\infty} |< x, A^* x_n^* >| \right), \quad \forall x \in X.\end{aligned}$$

But $(A^*x_n^*) \in \ell^1 < X^* >$ . Thus $(A^*x_n^*) \in F$. Now let $(z_n^*) \in F$. Then $\forall x \in X$,

$$\|TA(x)\| \leq \left( \sum_1^\infty |< x, z_n^* >| \right).$$

But $A^{-1}$ is an isometric onto operator. Thus $\forall x \in X,\ x = \ A^{-1}y$ for some $y \in X$. Then

$$\begin{aligned} \|Tx\| &= \left\|TA\left(A^{-1}y\right)\right\| \\ &\leq \left( \sum_1^\infty |< A^{-1}y, z_n^* >| \right) \\ &\leq \left( \sum_1^\infty |< y, \left(A^{-1}\right)^* z_n^* >| \right). \end{aligned}$$

Therefore $\left(\left(A^{-1}\right)^* z_n^*\right) \in E$. Thus $(z_n^*) \in F$ iff $z_n^* = (A^*x_n^*)$ for some $(x_n^*) \in E$. But

$$\|(z_n^*)\|_{\varepsilon(1)} = \|(A^*x_n^*)\|_{\varepsilon(1)} = \|(x_n^*)\|_{\varepsilon(1)}.$$

Consequently,

$$\|J(T)\|_{D(1)} = \|TA\|_{D(1)} = \inf_{(z_n^*)\in F} \|(z_n^*)\|_{\varepsilon(1)} = \inf_{(x_n^*)\in E} \|(x_n^*)\|_{\varepsilon(1)} = \|T\|_{D(1)},$$

and $J$ is an isometry.

Let $S \in D_1(X)$. Then $T = SA^{-1} \in D_1(X)$ being an operator ideal, and $J(T) = (SA^{-1})\ A = S$. ■

We end this section with the following questions:

**Question 1.** Is it true that if $J : \pi_p(X) \to \pi_p(X)$ is an isometric onto operator, then for all $T \in \pi_p(X), J(T) = ATB$ where $A, B \in L(X)$ are isometric onto operators.

**Question 2.** Is it true that if $J : D_1(X) \to D_1(X)$ is an isometric onto operator, then for all $T \in D_1(X),\ J(T) = ATB$ where $A, B \in L(X^{**})$ are isometric onto operators.

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