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The Category VRel(H)

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Abstract

We introduce the new category **VRel(H)** consisting of H-fuzzy relation spaces and H-fuzzy mappings between them satisfying a certain condition, where the concept of H-fuzzy mapping is the modification of one of fuzzy mapping introduced by Demirci[6]. And we investigate **VRel(H)** in the sense of a topological universe and show that **VRel(H)** is Cartesian closed over **Set**. Moreover, we construct the category **VFRel(H)** consisting of all H-fuzzy relational spaces over H-fuzzy sets and relation preserving mappings between them, and we find some properties of the category **VFRel(H)**. And we study the relations between the categories **Rel(H)** and **VRel(H)**.

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1. Introduction

Nel[18] introduced the concept of a topological universe which implies concrete guasitopos[1]. Thus every topological universe satisfies all the conditions

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of a topos except one condition of the subobject classifier. The notion of a topological universe has already been put to effective use in several areas of mathematics[16,17,19].

Zadeh[21] introduced the notion of a fuzzy relation naturally as a generalization of crisp relations in fuzzy set theory. After that time, Cerruti[4] made categories of L-fuzzy relations and studies their some properties. In particular, Hur[11] introduced the category **Rel(H)** consisting of H-fuzzy relations and proved that **Rel(H)** is topological universe and Cartesian closed.

Up to now, almost all the researchers studying the categories of fuzzy relations have used morphisms between fuzzy relations as crisp mappings satisfying a certain condition. Recently, Demirci[6] introduced the notion of fuzzy mappings and obtained many results. In particular, Hur et al.[12] studied relations between a fuzzy mapping and a fuzzy equivalence relation. Furthermore, also Hur et al.[13] investigated the category **VSet(H)** consisting of H-fuzzy spaces and H-fuzzy mappings between them satisfying a certain condition. In this paper, we introduce the new category **VRel(H)** consisting of H-fuzzy spaces and H-fuzzy mappings between them satisfying a certain condition, where the concept of H-fuzzy mapping is the modification of one of fuzzy mapping introduced by Demirci[6]. And we investigate **VRel(H)** in the sense of a topological universe and show that **VRel(H)** is Cartesion closed over **Set**. And we study the relations between the categories **Rel(H)** and **VRel(H)**

2. Preliminaries

In this section, we will introduce some basic definitions and well-known results from[2,8,15,18,20] which are needed in the next section.

Definition 2.1[15]. Let $\mathbf{A}$ be a concrete category and $((Y_\alpha, \xi_\alpha))_\Gamma$ a family of objects in $\mathbf{A}$ indexed by a class Γ. For any set X, let $(f_\alpha : X \to Y_\alpha)_\Gamma$ be a source of maps indexed by Γ. An $\mathbf{A}$-structure ξ on X is called *initial with respect to* $(X, (f_\alpha), ((Y_\alpha, \xi_\alpha)))$ provided that the following conditions hold :

(1) For each $\alpha \in \Gamma$, $f_\alpha : (X, \xi) \to (Y_\alpha, \xi_\alpha)$ is an $\mathbf{A}$-morphism.

(2) If (Z, ρ) is an $\mathbf{A}$-object and $g : Z \to X$ is a map such that for each $\alpha \in \Gamma$, the map $f_\alpha \circ g : (Z, \rho) \to (Y_\alpha, \xi_\alpha)$ is an $\mathbf{A}$-morphism, then $g : (Z, \rho) \to (X, \xi)$

is an **A**-morphism. In this case, $(f_\alpha : (X,\xi) \to (Y_\alpha, \xi_\alpha))_\Gamma$ is called an *initial source in* **A**.

Dual notions : *final structure ; final sink.*

Definition 2.2[15]. A concrete category **A** is called *topological over* **Set** provided that for each set X, for any family $((Y_\alpha, \xi_\alpha))_\Gamma$ of **A**-objects, and for any source $(f_\alpha : X \to Y_\alpha)_\Gamma$ of maps, there exists a unique **A**-structure ξ on X which is initial with respect to $(X, (f_\alpha), ((Y_\alpha, \xi_\alpha)))$.

Dual notion : *cotopological category.*

Result 2.A[15, Theorem 1.5]. *A concrete category **A** is topological if and only if **A** is cotopological.* .

Result 2.B[15, Theorem 1.6; 9, Proposition in Section 1]. *Let **A** be a topological category over* **Set.** *Then **A** is complete and cocomplete.*

Definition 2.3[15]. Let **A** be a concrete category.

(1) The **A**-*fibre* of a set X is the class of all **A**-structure on X.

(2) **A** is called *properly fibred over* **Set** provided that the following conditions hold :

(i) (*Fibre-smallness*) For each set X, the **A**-fibre of X is a set.

(ii) (*Terminal separator property*) For each singleton set X, the **A**-fibre of

X has precisely one element.

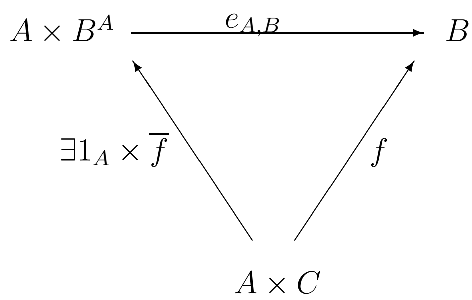
(iii) If ξ η and are **A**-structures on a set X such that $1_X : (X,\xi) \to (X,\eta)$ and $1_X : (X,\eta) \to (X,\xi)$ are **A**-morphisms, then $\xi = \eta$.

Definition 2.4[8]. A category **A** is called *Cartesian closed* provided that the following conditions hold :

(1) For any **A**-objects A and B, there exists a product $A \times B$ in **A**.

(2) Exponential objects exist in **A**, i.e., for any **A**-object A, the functor $A \times - : \mathbf{A} \to \mathbf{A}$ has a right adjoint, i.e., for any **A**-object B, there exist an **A**-object B^A and a **A**-morphism $e_{A,B} : A \times B^A \to B$(called the *evaluation*)

such that for any **A**-object C and any **A**-morphism $f : A \times C \to B$, there exists a unique **A**-morphism $\overline{f} : C \to B^A$ such that the diagram



commutes.

Definition 2.5[18]. A category **A** is called a *topological universe over* **Set** provided that the following conditions hold :

(1) **A** is well-structured over **Set**, i.e., (i) **A** is a concrete category ; (ii) **A** has the fibre-smallness condition ; (iii) **A** has the terminal separator property.

(2) **A** is cotopological over **Set**.

(3) Final episinks in **A** are preserved by pullbacks, i.e., for any final episink $(g_\lambda : X \to Y)_\Gamma$ and any **A**-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Gamma$, obtained by taking the pullback of f and g_λ, for each λ, is again a final episink.

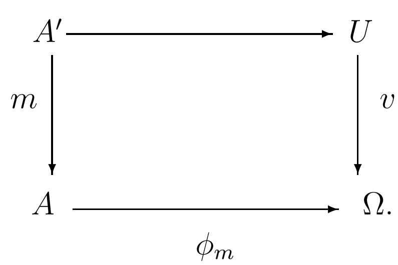
Definition 2.6[20]. A category **A** is called a *topos* provided that the following conditions hold:

(1) There is a terminal object U in **A**.

(2) **A** has equalizers.

(3) **A** is Cartesian closed.

(4) There is a subobject classifier in **A**, i.e., there is an object Ω and morphism v from U to Ω such that for each monomorphism m from A' to A, there exists a unique morphism $\varnothing_m$ from A to Ω such that the following diagram is a pullback:



Remark. Let **A** be any category with a subobject classifier. If f is any biomorphism in **A**, then f is an isomorphism in **A**(cf. [3]).

Definition 2.7[2]. A lattice H is called a *complete Heyting algebra,* if H satisfies the following conditions hold :

(1) H is a complete lattice.

(2) For any $a, b \in H$, the set $\{x \in H : x \wedge a \leq b\}$ has a greatest element denoted by $a \rightarrow b$ (called *the relative pseudo - complement of a in b)*, i.e., $x \wedge a \leq b$ if and only if $x \leq (a \rightarrow b)$.

In particular, if H is a complete Heyting algebra with the least element 0, then for each $a \in H,\ N(a) = a \rightarrow 0$ ia called the *negation* or the *pseudo - complement* of a.

Throughout this paper, we will use H as a complete Heyting algebra with the least element 0 and the largest element 1.

3. The category VRel(H)

In this section, we introduce the category **VRel(H)** of fuzzy relational spaces and show that it has structures similar to those of **VSet(H)** (see [13]).

Definition 3.1[6,13]. A mapping $E_X : X \times X \longrightarrow H$ is called *an H-fuzzy equality on* X if it satisfies the following conditions:

(i) $E_X(x, y) = 1 \ \Leftrightarrow x = y \ \forall x, y \in X$,

(ii) $E_X(x, y) = E_X(y, x) \ \forall x, y \in X$,

(iii) $E_X(x, y) \wedge E(y, z) \leq E_X(x, z) \ \forall x, y, z \in X$.

We will denote the set of all H-fuzzy equalities on X as $E_H(X)$.

Definition 3.2[6,13]. A H-fuzzy relation f on $X \times Y$ is called an *H-fuzzy mapping w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$*, denoted by $f : X \longrightarrow Y$, if it satisfies the following conditions:

(i) $\forall x \in X,\ \exists y \in Y$ such that $f(x, y) > 0$,

(ii) $\forall x, y \in X,\ \forall z, w \in y,\ f(x, z) \wedge f(y, w) \wedge E_X(x, y) \leq E_Y(z, w)$.

Definition 3.3[6,13]. The *identity H-fuzzy mapping* I_X on X is a H-fuzzy relation on $X \times X$ defined by

$$I_X(x,y) = \begin{cases} 1, & \text{if x=y,} \\ 0, & \text{if x}\neq \text{ y, } \ \forall \text{x,y} \in \text{X.} \end{cases}$$

It is clear that $I_X \in E_H(X)$. Also, if $f : X \to Y$ is an (ordinary) mapping, then it is an H-fuzzy mapping w.r.t. $I_X \in E_H(X)$ and $I_Y \in E_H(Y)$.

Definition 3.4[6,13]. Let $f : X \to Y$ be an H-fuzzy mapping w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$. Then f is said to be:

(i)*strong* if $\forall x \in X$, $\exists y \in Y$ such that $f(x,y) = 1$,

(ii)*surjective* if $\forall y \in Y$, $\exists x \in X$ such that $f(x,y) > 0$,

(iii)*strong surjective* if $\forall y \in Y$, $\exists x \in X$ such that $f(x,y) = 1$,

(iv)*injective* if $f(x,z) \wedge f(y,w) \wedge E_Y(z,w) \leq E_X(x,y)$, $\forall x,y \in X$, $\forall z,w \in Y$,

(v)*bijective* if it surjective and injective,

(vi)*strong bijective* if it is strong surjective and injective.

In particular, if $f(x,y) = 1$, then we will write $y = f(x)$. It is clear that I_X is a strong H-fuzzy mapping w.r.t. $E_X \in E_H(X)$. Moreover, it is strong bijective w.r.t. $E_X \in E_H(X)$.

Definition 3.5[11]. Let X be a set. R is called an *H-fuzzy relation* (or simply, a *fuzzy relation*) on X if $\mu_R : X \times X \to H$ is a mapping. In this case, (X,R) is called an *H-fuzzy relational space* (or, simply, a *fuzzy relational space*).

Definition 3.6[6,13]. Let R and S be H-fuzzy relations on $X \times Y$ and $Y \times Z$, respectively. Then

(i)*the sup-min composition of R and S*, denoted by $S \circ R$, is a H-fuzzy relation on $X \times Z$ defined by

$$S \circ R(x,z) = \bigvee_{y \in Y} [R(x,y) \wedge S(y,z)] \ \forall x \in X, \ \forall z \in Z,$$

(ii)*the inverse of R*, denoted by R, is a H-fuzzy relation on $Y \times X$ defined by

$$R^{-1}(y,x) = R(x,y),\ \forall x \in X,\ \forall y \in Y.$$

Result 3.A[13, Proposition 3.6]. Let $f : X \to Y$ and $g : Y \to Z$ be H-fuzzy mapping w.r.t. $E_X \in E_H(X)$, $E_Y \in E_H(Y)$ and $E_Z \in E_H(Z)$. Then the sup-min composition $g \circ f$ is an H-fuzzy mapping $g \circ f : X \to Z$ w.r.t. $E_X \in E_H(X)$ and $E_Z \in E_H(Z)$.

Result 3.B[13, Corollary 3.6]. Let $f : X \to Y$ and $g : Y \to Z$ be H-fuzzy mappings w.r.t. $E_X \in E_H(X)$, $E_Y \in E_H(Y)$ and $E_Z \in E_H(Z)$. If f and g are strong [resp. surlective, strong surjective, injective, bijective and strong bijective], then $g \circ f$ is strong [resp. surjective, strong surjective, injective, bijective and strong bijective].

Definition 3.7[13]. Let $f : X \to Y$ be an H-fuzzy mapping w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$, let $A \in H^X$ and let $B \in H^X$.

(i) The *image of A under f*, denoted by $f(A)$, is an H-fuzzy set in Y defined as follows:

$$f(A)(y) = \bigvee_{x \in X} [A(x) \wedge f(x,y)] \quad \forall y \in Y.$$

(ii) The *preimage of B under f*, denoted by $f^{-1}(B)$, is an H-fuzzy set in X defined as follows:

$$f^{-1}(B)(x) = \bigvee_{y \in Y} [B(y) \wedge f(x,y)] \quad \forall x \in X.$$

Definition 3.8[13, Proposition 3.10]. Let $f : X \to Y$ be an H-fuzzy mapping $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$. Then $f^2 = f \times f : X \times X \to Y \times Y$ is called the *fuzzy product mapping of f w.r.t.* $E_{X\times X} = E_X \times E_X \in E_H(X \times X)$ *and* $E_{Y \times Y} = E_Y \times E_Y \in E_H(Y \times Y)$ if $f^2 : (X \times X) \times (Y \times Y) \to H$ is the mapping defined as follows:

$$f^2((x,x'),(y,y')) = f(x,y) \wedge f(x',y'),\ \ \forall (x,x') \in X \times X,\ \ \forall (y,y') \in Y \times Y.$$

Definition 3.9. Let (X, R_X) and (Y, R_Y) be H-fuzzy relational spaces and let $f : X \to Y$ be an H-fuzzy mapping w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$.

Then $f : (X, R_X) \to (Y, R_Y)$ is called a *relation preserving mapping* if $R_X \subset f^{-1}(R_Y)$, where $f^{-2} = (f \times f)^{-1}$. In particular, a relation preserving mapping $f : (X, R_X) \to (Y, R_Y)$ is called an *epimorphism* [resp. a *monomorphism*, an *isomorphism*] if it is surjective [resp. injective and bijective].

The following is the immediate result of Result 3.A and Definitions 3.7, 3.8 and 3.9.

Proposition 3.10. Let (X, R_X), (Y, R_Y) and (Z, R_Z) be H-fuzzy relational spaces, and let $f : X \to Y$ and $g : Y \to Z$ be H-fuzzy mappings w.r.t. $E_X \in E_H(X)$, $E_Y \in E_H(Y)$ and $E_Z \in E_H(Z)$.

(a) The identity H-fuzzy mapping $I_X : (X, R_X) \to (X, R_X)$ w.r.t. $E_X \in E_H(X)$ is a relation preserving mapping.

(b) If $f : (X, R_X) \to (Y, R_Y)$ and $g : (Y, R_Y) \to (Z, R_Z)$ are relation preserving mappings, then $g \circ f : (X, R_X) \to (Z, R_Z)$ is a relation preserving mapping.

From Result 3.B and Proposition 3.10, we can form the concrete category **VRel(H)** consisting of H-fuzzy relational spaces and strong relation preserving mappings between them. Every **VRel(H)** strongmorphism will be called a **VRel(H)**-*mapping*.

Lemma 3.11. The category **VRel(H)** is topological over **Set**.

Proof. Let X be any set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of H-fuzzy relational spaces indexed by a class Γ. Suppose $(f_\alpha : X \to X_\alpha)_\Gamma$ is a source of strong H-fuzzy mappings w.r.t. $E_X \in E_H$ and $E_{X_\alpha} \in E_H(X_\alpha)$. Define $R_X : X \times X \to H$ by $R_X(R)(x, x') = [\bigcap_{\alpha \in \Gamma} f_\alpha^{-2}(R_\alpha)](x, x')$, $\forall (x, x') \in X \times X$. Then clearly (X, R_X) is a H-fuzzy relational space and each $f_\alpha : (X, R_X) \to (X_\alpha, R_\alpha)$ is a **VRel(H)**-mapping. Let (Y, R_Y) be any H-fuzzy relational space and suppose $g : Y \to X$ is any strong H-fuzzy mapping w.r.t. $E_Y \in E_H(Y)$ and E_X for which $f_\alpha \circ g : (Y, R_Y) \to (X_\alpha, R_\alpha)$ is a **VRel(H)**-mapping for each $\alpha \in \Gamma$. Then, for each $\alpha \in \Gamma$

$$R_Y \subset (f_\alpha \circ g)^{-2}(R_\alpha) = g^{-2}(f_\alpha^{-2}(R_\alpha)) = g^{-2}(f_\alpha^{-2}(R_\alpha)).$$

Thus $R_Y \subset g^{-2}(\bigcap_{\alpha\in\Gamma} f^{-2}(R_\alpha)) = g^{-2}(R_X)$. So $g : (Y, R_Y) \to (X, R_X)$ is a **VRel(H)**-mapping. Hence $(f_\alpha : (X, A_X) \to (X, A_\alpha))_\Gamma$ is an initial source in **VRel(H)**. This completes the proof. □

Example 3.11. (1) *The inverse image of a H-fuzzy relation structure.* Let X be a set, let (Y, R_Y) be a H-fuzzy relational space and let $f : X \to Y$ be a strong H-fuzzy mapping w.r.t. $E_X \in E_H$ and $E_Y \in E_H(Y)$. Then there exists a unique H-fuzzy relation R_X in X for which $f : (X, R_X) \to (Y, R_Y)$ is a **VRel(H)**-mapping. In fact, $R_X = f^{-2}(R_Y)$. In this case, R_X is called the *inverse image under f of the H-fuzzy relation structure R_Y*.

(2) *The H-fuzzy product structure.* Let $((X_\alpha, R_\alpha))_\Gamma$ be any family of H-fuzzy spaces, let $X = \prod_{\alpha\in\Gamma} X_\alpha$ and for each $\alpha \in \Gamma$, let $\mathrm{pr}_\alpha : X \to X_\alpha$ be the H-fuzzy projection w.r.t. $E_X \in E_H(X)$ and $E_{X_\alpha} \in E_H(X_\alpha)$. Then there exists a unique H-fuzzy relation structure R_X in X w.r.t. $(X, (\mathrm{pr}_\alpha)_{\alpha\in\Gamma}, ((X_\alpha, R_\alpha))_{\alpha\in\Gamma})$. In this case, R_X is called *the H-fuzzy product of H-fuzzy relation structures* in the X_α and denoted by $R_X = \prod_{\alpha\in\Gamma} R_\alpha$, and $(\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma} R_\alpha)$ is called *the H-fuzzy product relational space* of $((X_\alpha, R_\alpha))_\Gamma$. In fact, $\prod_{\alpha\in\Gamma} R_\alpha = \bigcap_{\alpha\in\Gamma} \mathrm{pr}_\alpha^{-2}(R_\alpha)$. In particular, if $\Gamma = \{1, 2\}$, then $(R_1 \times R_2)((x_1, y_1), (x_2, y_2)) = R_1(x_1, x_2) \wedge R_2(y_1, y_2)$ for any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$.

The following is the immediate result of Lemma 3.11 and Result 2.B

Corollary 3.11. The category **VRel(H)** is complete and cocomplete.

It is well-known[15] that a category is topological if and only if it is cotopological. However, we show directly that **VRel(H)** is cotopological.

Lemma 3.12. The category **VRel(H)** is cotopological over **Set**.

Proof. Let X be any set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of H-fuzzy relational spaces indexed by a class Γ. Suppose $(f_\alpha : X_\alpha \to X)_\Gamma$ is a sink of strong H-fuzzy mappings w.r.t. $E_{X_\alpha} \in E_H(X_\alpha)$ and $E_X \in E_H(X)$. We define

$R_X : X \times X \to H$ by $R_X = \bigcup_{\alpha\in\Gamma} f_\alpha^2(R_\alpha)$. Then clearly R_X is well-defined and each $f_\alpha : (X_\alpha, R_\alpha) \to (X, R_X)$ is a **VRel(H)**-mapping. For each H-fuzzy relational space (Y, R_Y), let $g : X \to Y$ be a strong H-fuzzy mapping w.r.t. E_X and $E_Y \in E_H(Y)$ such that each $g \circ f_\alpha : (X_\alpha, R_\alpha) \to (Y, R_Y)$ is a **VRel(H)**-mapping. Then $R_\alpha \subset (g \circ f_\alpha)^{-2}(R_Y), \quad \forall \alpha \in \Gamma$. Thus, for each $(x_\alpha, x'_\alpha) \in X_\alpha \times X_\alpha$,

$$\begin{aligned} R_\alpha(x_\alpha, x'_\alpha) &\le (g \circ f_\alpha)^{-2}(R_Y)(x_\alpha, x'_\alpha) \\ &= f_\alpha^{-2}(g^{-2}(R_Y))(x_\alpha, x'_\alpha) \\ &= \bigvee_{(x,x')\in X\times X} [g^{-2}(R_Y)(x, x') \wedge f_\alpha(x_\alpha, x) \wedge f_\alpha(x'_\alpha, x')] \\ &\le \bigvee_{(x,x')\in X\times X} g^{-2}(R_Y)(x, x'), \end{aligned}$$

i.e., $R_\alpha(x_\alpha, x'_\alpha) \le g^{-2}(R_Y)(x, x'), \quad \forall (x, x') \in X \times X$. So

$$R_X(x, x') = [\bigcup_{\alpha\in\Gamma} f_\alpha^2(R_\alpha)](x, x') \quad \le g^{-2}(R_Y)(x, x') \ \ \forall (x, x') \in X \times X.$$

i.e., $R_X \subset g^{-2}(R_Y)$. Hence $g : (X, R_X) \to (Y, R_Y)$ is a **VRel(H)**-mapping. Therefore **VRel(H)** is cotopological over **Set**. □

Result 3.C[13, Proposition 3.11]. Let $f : X \to Y$ be a strong H-fuzzy mapping w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$, and let $g : Z \to Y$ be a strong H-fuzzy mapping w.r.t. $E_Z \in E_H(Z)$ and E_Y. Let $U = \{(x, z) \in X \times Z : \exists y \in Y \text{ such that } f(x, y) = 1 = g(z, y)\}$. Then the restriction $E_U = (E_X \times E_Z) \mid_{U\times U}: U \times U \to H$ is a H-fuzzy equality on U. Moreover, $\mathrm{pr}_X : U \to X$ and $\mathrm{pr}_Z : U \to Z$ are projections w.r.t. E_U and E_X, and E_U and E_Z, respectively.

Lemma 3.13. Final episinks in **VRel(H)** are preserved by pullbacks.

Proof. Let $(g_\alpha : (X_\alpha, R_\alpha) \longrightarrow (Y, R_Y))_\Gamma$ be any final episink in **VRel(H)** w.r.t. $E_\alpha \in E_H(X_\alpha)$ and $E_Y \in E_H(Y)$, and let $f : (W, R_W) \longrightarrow (Y, R_Y)$ be any **VRel(H)**-mapping w.r.t. $E_W \in E_H(W)$ any E_Y. For each $\alpha \in \Gamma$, let

$$U_\alpha = \{(w, x_\alpha) \in W \times X_\alpha : \exists y \in Y \text{ such that } f(w, y) = 1 = g_\alpha(x_\alpha, y)\}$$

and let $R_{U_\alpha} = (R_W \times R_\alpha) \mid_{U_\alpha\times U_\alpha}$. Then clearly (U_α, R_{U_α}) is a H-fuzzy relational space. By Result 3.C, for each $\alpha \in \Gamma$, $e_\alpha : U_\alpha \to W$ and $p_\alpha : U_\alpha \to X_\alpha$ are projections of U_α w.r.t. E_{U_α} and E_W, and E_{U_α} and E_X, respectively. Furthermore, for each $\alpha \in \Gamma$, $e_\alpha : (U_\alpha, R_{U_\alpha}) \to (W, R_W)$ and $p_\alpha : (U_\alpha, R_{U_\alpha}) \to (X_\alpha, R_\alpha)$

are **VRel(H)**-mappings and the following diagram is a pullback square in **VRel(H)**:

$$\begin{array}{ccc} (U_\alpha, R_{U_\alpha}) & \xrightarrow{\quad p_\alpha \quad} & (X_\alpha, R_\alpha) \\ e_\alpha \downarrow & & \downarrow g_\alpha \\ (W, R_W) & \xrightarrow[\quad f \quad]{} & (Y, R_Y). \end{array}$$

Let $w \in W$. Since $f : W \to X$ is a strong H-fuzzy mapping, $\exists y_\circ \in Y$ such that $f(w, y_\circ) = 1$. Since $(g_\alpha)_\Gamma$ is a final episink, for each $\alpha \in \Gamma$, and for $y_\circ \in Y$,

$$\exists x_{\alpha_\circ} \in X_\alpha \text{ such that } g_\alpha(x_{\alpha_\circ}, y_\circ) = 1.$$

Thus $(w, x_{\alpha_\circ}) \in U_\alpha$ and $e_\alpha((w, x_{\alpha_\circ}), w) = 1$. So $(e_\alpha)_\Gamma$ is an episink in **VRel(H)**. Moreover $(e_\alpha)_\Gamma$ is final. : Let R^*_W be the final structure on W w.r.t. $(e_\alpha)_\Gamma$ and let $(w, w') \in W \times W$. Then

$$\begin{aligned} R_W(w, w') &= R_W(w, w') \wedge R_W(w, w') \\ &\le R_W(w, w') \wedge f^{-2}(R_Y)(w, w') \\ &\qquad [\text{ Since } f : (W, R_W) \to (Y, R_Y) \text{ is a } \mathbf{VRel(H)}\text{-mapping }] \\ &= R_W(w, w') \wedge \Big(\bigvee_{(y,y') \in Y \times Y} [R_Y(y, y') \in Y \times Y \wedge f(w, y) \wedge f(w', y')]\Big) \\ &= R_W(w, w') \wedge \Big(\bigvee_{(y,y') \in Y \times Y} \bigvee_{\alpha \in \Gamma} [g^2_\alpha(R_\alpha)(y, y') \wedge f(w, y) \wedge f(w', y')]\Big) \\ &\qquad [\text{ Since } (g_\alpha)_\Gamma \text{ is final }] \\ &= R_W(w, w') \wedge \Big(\bigvee_{(y,y') \in Y \times Y} \bigvee_{\alpha \in \Gamma} \bigvee_{(x_\alpha, y_\alpha) \in X_\alpha \times X_\alpha} [R_\alpha(x_\alpha, y_\alpha) \wedge g_\alpha(x_\alpha, y) \\ &\qquad \wedge g_\alpha(y_\alpha, y') \wedge f(w, y) f(w', y')]\Big) \\ &= \bigvee_{\alpha \in \Gamma} \bigvee_{(x_\alpha, y_\alpha) \in X_\alpha \times X_\alpha} [R_W(w, w') \wedge R_\alpha(x_\alpha, y_\alpha) \wedge \Big(\bigvee_{(y,y') \in Y \times Y} g_\alpha(x_\alpha, y) \\ &\qquad \wedge g_\alpha(y_\alpha, y') \wedge f(w, y) \wedge f(w', y')\Big)] \\ &= \bigvee_{\alpha \in \Gamma} \bigvee_{((w,x_\alpha),(w',y_\alpha)) \in U_\alpha \times U_\alpha} [R_{U_\alpha}(w, x_\alpha) \wedge R_{U_\alpha}(w', y_\alpha) \wedge e_\alpha((w, x_\alpha) \\ &\qquad , w) \wedge e_\alpha((w', y_\alpha), w')] \ [\text{ Since } R_{U_\alpha} = (R_W \times R_\alpha) \mid_{U_\alpha \times U_\alpha}, \end{aligned}$$

f is

strong and g is strong surjective]

$$= R^*_W(w).$$

Thus $R_W \subset R^*_W$. On the other hard, since $(e_\alpha : (U_\alpha, R_{U_\alpha}) \to (W, R^*_W))_\Gamma$ is final, $I_W : (W, R^*_W) \to (W, R_W)$ is a **VRel(H)**-mapping. So $R^*_W \subset R_W$. Hence $A_W = A^*_W$. This completes the proof. □

For any singleton set $\{a\}$, since the H-fuzzy relation structure $R_{\{a\}}$ on $\{a\}$ is not unique, the category **VRel(H)** is not properly fibred over **Set**. Hence, by Lemmas 3.11 and 3.13, we obtain the following result.

Theorem 3.14. The category **VRel(H)** satisfies all the conditions of a topological universe over **Set** except the terminal separator property.

Theorem 3.15. The category **VRel(H)** is Cartesian closed over **Set**.

Proof. It is obvious that **VRel(H)** has products by Corollary 3.11. Then it is sufficient to show that **VRel(H)** has exponential objects.

For any H-fuzzy spaces $\mathbf{X} = (X, R_X)$ and $\mathbf{Y} = (Y, R_Y)$, let Y^X be the set of all strong H-fuzzy mappings from X to Y. We define a mapping $R_{Y^X} : Y^X \times Y^X \to H$ as follows: for each $(f, g) \in Y^X \times Y^X$,

$$\begin{aligned} R_{Y^X}(f,g) &= \bigvee\{h \in H : R_X \cap \underline{h} \subset (f^{-1} \times g^{-1})(R_Y)\} \\ &= \bigvee\{h \in H : R_X(x,x') \wedge h \leq \bigvee_{(y,y')\in Y\times Y} [R_Y(y,y') \wedge f(x,y) \\ &\qquad \wedge g(x',y')], \forall (x,x') \in X \times X\} \end{aligned}$$

where $\underline{h}(x) = h, \ \ \forall x \in X$. Since f and g are strong,

$$R_{Y^X}(f,g) = \bigvee\{h \in H : R_X(x,x') \wedge h \leq \bigvee_{f(x,y)=1, g(x',y')=1} R_Y(y,y')\}.$$

Then clearly $(Y^X, R_{Y^X}) \in$ **VRel(H)**. Let $\mathbf{Y}^X = (Y^X, R_{Y^X})$. Then, by the definition of R_{Y^X},

$$R_X(x,x') \wedge R_{Y^X}(f,g) \leq \bigvee_{f(x,y)=1, g(x',y')=1} R_Y(x,y), \ \ \forall (f,g) \in Y^X \times Y^X,$$

$\forall (x,x') \in X \times X$.

We define a mapping $e_{X,Y} : (X \times Y^X) \times Y \to H$ by

$$e_{X,Y}((x,f),y) = f(x,y) \ \ \forall (x,f) \in X \times Y^X, \ \ \forall y \in Y.$$

Then clearly $e_{X,Y}$ is an H-fuzzy relation on $(X \times Y^X) \times Y$. Now we define a mapping $E_{X \times Y^X}$: $(X \times Y^X) \times (X \times Y^X) \to H$ as follows : For any $(x,f), \ (x',g) \in X \times Y^X$,

$$E_{X\times Y^X}((x,f), \ (x',g)) = (\bigwedge_{y\in Y} f(x',y) \wedge \bigwedge_{y'\in Y} g(x,y')) \wedge E_X(x,x') \wedge$$

$E'_X(x,x')$,

where $f : X \to Y$ and $g : X \to Y$ are strong H-fuzzy mappings w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$, and $E'_X \in E_H(X)$ and $E'_Y \in E_H(Y)$, respectively.

Then, by the process of the proof of Theorem 4.8 in [13], $e_{X,Y} : X \times Y^X \to Y$ is a strong H-fuzzy mapping w.r.t. $E_{X \times Y^X}$ and $E \in E_H(Y)$, where $E = E_Y \times E'_Y$ is an H-fuzzy equality on Y. Let $((x,f),(x',g)) \in (X \times Y^X) \times (X \times Y^X)$. Then

$$\begin{aligned} e_{X,Y}{}^{-2}(R_Y)((x,f),(x',g)) &= \bigvee_{(y,y') \in Y \times Y} [R_Y(y,y') \wedge e_{X,Y}((x,f),y) \\ &\qquad \wedge e_{X,Y}((x',g),y')] \\ &= \bigvee_{(y,y') \in Y \times Y} [R_Y(y) \wedge f(x,y) \wedge g(x',y')] \\ &= \bigvee_{f(x,y)=1,\ g(x',y')=1} R_Y(y,y') \\ &\quad [\text{ Since } f \text{ and } g \text{ are strong }] \\ &\geq R_X(x,x') \wedge R_{Y^X}(f,g) \\ &= (R_X \times R_{Y^X})((x,f),(x',g)). \end{aligned}$$

Thus $R_X \times R_{Y^X} \subset e_{X,Y}{}^{-2}(R_Y)$. So $e_{X,Y} : \mathbf{X} \times \mathbf{Y}^X \to \mathbf{Y}$ is a **VRel(H)**-mapping.

For any $\mathbf{Z} = (Z, R_Z) \in$ **VRel(H)**, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be a **VRel(H)**-mapping w.r.t. $E_{X \times Z} = E_X \times E_Z \in E_H(X \times Z)$ and $E_Y \in E_H(Y)$. We define a mapping $\bar{h} : Z \times (X \times Y) \to H$ as follows: $\forall z \in Z, \ \ \forall x \in X, \ \ \forall y \in Y$,

$$\bar{h}(z)(x,y) = h((x,z),y), \text{ where } \bar{h}(z)(x,y) = \bar{h}(z,(x,y)).$$

Since h is strong, it is clear that $\bar{h}(z)$ is strong. Thus $\bar{h}(z) \in Y^X, \forall z \in Z$. So $\bar{h} : Z \to Y^X$ is a strong H-fuzzy mapping. Let $z, z' \in Z$ and let $x, x' \in X$. Then

$$\begin{aligned} R_X(x,x') \wedge R_Z(z,z') &= (R_X \times R_Z)((x,z),(x',z')) \\ &\leq h^{-2}(R_Y)((x,z),(x',z')) \\ &\quad [\text{ Since } h : X \times Z \to Y \text{ is a } \mathbf{VRel(H)}\text{-mapping }] \\ &= \bigvee_{(y,y') \in Y \times Y} [R_Y(y,y') \wedge h((x,z),y) \wedge h((x',z'),y')] \\ &= \bigvee_{(y,y') \in Y \times Y} [R_Y(y,y') \wedge \bar{h}(z)(x,y) \wedge \bar{h}(z')(x',z')] \\ &= \bigvee_{\bar{h}(z)(x,y)=1,\ \bar{h}(z)(x',y')=1} R_Y(y,y'). \\ &\quad [\text{ Since } \bar{h}(z) \text{ is strong }] \end{aligned}$$

Thus, by the definition of R_{Y^X},

$$R_Z(z) \leq \bar{h}^{-2}(R_{Y^X})(z).$$

So $R_Z \subset \bar{h}^{-2}(R_{Y^X})$. Hence $\bar{h} : \mathbf{Z} \to \mathbf{Y}^X$ is a **VRel(H)**-mapping. Moreover, $\bar{h}$ is unique **VRel(H)**-mapping such that $e_{X,Y} \circ (I_X \times \bar{h}) = h$. This completes the proof. □

4. The category VFRel(H)

Definition 4.1[13]. The concrete category **VSet(H)** is defined by: Objects (X, A_X), called an H-fuzzy space, where X is any set and $A_X \in H^X$. A morphism $f : (X, A_X) \to (Y, A_Y)$ is a strong H-fuzzy mapping w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$ satisfying $A_X \leq f^{-1}(A_Y)(x), \ \forall x \in X$. Every **VSet(H)**-morphism is called *a **VSet(H)**-mapping.*

Definition 4.2 [10]. *An H-fuzzy relation R on* an H-fuzzy space (X, A_X) is an H-fuzzy set in $X \times X$ satisfying $R(x, y) \leq A_X(x) \wedge A_X(y)$ for any $x, y \in X$. In this case, the triple (X, A_X, R) is called *an H-fuzzy relational space over* (X, A_X).

Definition 4.3. An H-fuzzy mapping $f : (X, A_X, R_X) \to (Y, A_Y, R_Y)$ w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$ is called a *relation preserving mapping* if it satisfies the following conditions:

(i) $f : (X, A_X) \to (Y, A_Y)$ is a **VSet(H)**-mapping,

(ii) $f : (X, R_X) \to (Y, R_Y)$ is a **VRel(H)**-mapping.

We denote the category of all H-fuzzy relational spaces over H-fuzzy spaces and relation preserving strong H-fuzzy mappings between them by **VFRel(H)**, and the mixture of the categories **VSet(H)** and **VRel(H)** by **VSet(H)**∧**VRel(H)**(cf [14]). Since **VSet(H)** and **VRel(H)** are topological over **Set** by Lemma 4.4 in [13] and Lemma 3.11, so is the mixture **VSet(H)**∧**VRel(H)** with natural structures by Proposition 2 in [14].

Lemma 4.4. **VFRel(H)** is a bi(co)reflective subcategory of **VSet(H)**∧**VRel(H)**.

Proof. Let (X, A, R) be an object in **VSet(H)**∧**VRel(H).** We define a mapping $A_X : X \to H$ as follows: For each $x \in X$,

$$A_X(x) = A(x) \vee [\bigvee_{y \in X} R(x, y)].$$

Then it is easily seen that $I_X : (X, A, R) \to (X, A_X, R)$ is a **VFRel(H)**-reflection of (X, A, R). Now we define a mapping $R_X : X \times X \to H$ as follows: For any $x, y \in X$,

$$R_X(x, y) = R(x, y) \wedge A(x) \wedge A(y).$$

Then $I_X : (X, A, R_X) \to (X, A, R)$ is a **VFRel(H)**-coreflection of (X, A, R). This completes the proof. □

The following is the immediate result of Lemma 4.4 and Theorems 2.6 and 2.8 in[14].

Theorem 4.5. (a) The category **VFRel(H)** is topological over **Set.**

(b) Final episinks in **VFRel(H)** are preserved by pullbacks.

Remark 4.6. (a) Let X be a set and let $(f_\alpha : X \to (X_\alpha, A_\alpha, R_\alpha))_{\alpha\in\Gamma}$ be a source, where $(X_\alpha, A_\alpha, R_\alpha) \in$**VFRel(H)** for each $\alpha \in \Gamma$. We define two mappings $A_X : X \to H$ and $R_X : X \times X \to H$ as follows, respectively:

$$A(x) = \bigwedge_{\alpha\in\Gamma} f^{-1}(A_\alpha)(x),\ \forall x \in X$$

and

$$R_X(x, y) = \bigwedge_{\alpha\in\Gamma} f^{-2}(R_\alpha)(x, y),\ \forall x, y \in X.$$

Then (X, A_X, R_X) is equipped with the initial structure w.r.t. $(f_\alpha)_\Gamma$ in **VFRel(H)**.

(b) Let X be a set and let $(f_\alpha : (X_\alpha, A_\alpha, R_\alpha) \to X)_{\alpha\in\Gamma}$ be a sink, where $(X_\alpha, A_\alpha, R_\alpha) \in$**VFRel(H)**, $\forall\alpha \in \Gamma$. We define two mappings $A_X : X \to H$ and $R_X : X \times X \to H$ as follows, respectively:

$$A_X(x) = \bigvee_{\alpha\in\Gamma} f_\alpha(A_\alpha)(x),\ \forall x \in X$$

and

$$R_X(x, y) = \bigvee_{\alpha\in\Gamma} f_\alpha(x, y),\ \forall x, y \in X.$$

Then (X, A_X, R_X) is equipped with the final structure w.r.t. $(f_\alpha)_\Gamma$.

(c) Since both H-fuzzy set structures and H-fuzzy relational structures on a singleton set are not unique, **VFRel(H)** is not properly fibred.

(d) Let $(g_\alpha : (X_\alpha, A_\alpha, R_\alpha) \to (Y, A_Y, R_Y))_{\alpha\in\Gamma}$ be any final episink in **VFRel(H)** and $f : (W, A_W, R_W) \to (Y, A_Y, R_Y)$ be any H-fuzzy mapping w.r.t. $E_W \in E_H(W)$ and $E_Y \in E_H(Y)$ in **VFRel(H)**. For each $\alpha \in \Gamma$, let

$$U_\alpha = \{(w, x_\alpha) \in W\times X_\alpha : \exists y \in Y \text{ such that } f(w, y) = 1 = g_\alpha(x_\alpha, y)\},$$

let $A_{U_\alpha} = (A_W \times A_\alpha) \mid_{U_\alpha\times U_\alpha}$ and let $R_{U_\alpha} = (R_W \times R_\alpha) \mid_{U_\alpha\times U_\alpha}$. Then, for each $\alpha \in \Gamma$, $e_\alpha : (U_\alpha, A_{U_\alpha}, R_{U_\alpha}) \to (W, A_W, R_W)$ is the pullback of g_α along f in **VFRel(H)**, where $e_\alpha : U_\alpha \to W$ is the H-fuzzy projection of U_α w.r.t. $E_{U_\alpha} \in E_H(U_\alpha)$ and E_W. Moreover, $(e_\alpha : (U_\alpha, A_{U_\alpha}, R_{U_\alpha}) \to (W, A_W, R_W))_{\alpha\in\Gamma}$

is a final episink in **VFRel(H)**.

Remark 4.7. (a) The category **VFRel(H)** is topological over $VSet(H)$: Let (X, A_X) be any H-fuzzy space and let $((X_\alpha, A_\alpha, R_\alpha))_\Gamma$ be any family of H-fuzzy relational spaces. Let $(f_\alpha : (X, A_X) \to (X_\alpha, A_\alpha))_{\alpha\in\Gamma}$ be any source in **VSet(H)**. We define a mapping $R_X : X \times X \to H$ as follows: For any $x, y \in X$,

$$R_X(x, y) = (\bigwedge_{\alpha\in\Gamma} f_\alpha^{-2}(R_\alpha)(x, y)) \wedge A_X(x) \wedge A_X(y).$$

Then R_X is the initial structure on (X, A_X) w.r.t. $(f_\alpha)_\Gamma$.

(b) The category **VFRel(H)** is cotopological over **VSet(H)**: Let (X, A_X) be any H-fuzzy space and let $((X_\alpha, A_\alpha, R_\alpha))_\Gamma$ be any family of H-fuzzy relational spaces. Let $(f_\alpha : (X_\alpha, A_\alpha) \to (X, A_\alpha))_{\alpha\in\Gamma}$ be any sink in **VSet(H)**. We define a mapping $R_X : X \times X \to H$ as follows: For any $x, y \in X$,

$$R_X(x, y) = \bigvee_{\alpha\in\Gamma} f^2(R_\alpha).$$

Then R_X is the final structure on (X, A_X) w.r.t. $(f_\alpha)_\Gamma$.

Theorem 4.8. The category **VFRel(H)** is Cartesian closed.

Proof. Since **VFRel(H)** has finite products by Theorem 4.5, it is enough to show that **VFRel(H)** has exponetials.

For any H-fuzzy relational spaces $\mathbf{X} = (X, A_X, R_X)$ and $\mathbf{Y} = (Y, A_Y, R_Y)$, let Y_X be the set of all morphisms from X to Y in **VFRel(H)**. We define two mappings $A_{Y^X} : Y^X \to H$ and $R_{Y^X} : Y_X \times Y_X \to H$ as follows, respectively:

$$A_{Y^X}(f) = \bigvee\{h \in H : A_X(x) \wedge h \leq \bigvee_{y\in Y}[A_Y(y) \wedge f(x, y)], \forall x \in X\},\ \forall f \in Y^X$$

and

$$R_{Y^X}(f, g) = \bigvee\{h \in H : R_X(x, x') \wedge h \leq (\bigvee_{(y,y')\in Y\times Y}[R_Y(y, y') \wedge f(x, y) \wedge g(x', y')]) \wedge (A_{Y^X}(f) \wedge A_{Y^X}(g))\ \forall x, y \in X\}.$$

Then clearly R_{Y^X} is an H-fuzzy relation on (Y^X, A_{Y^X}). Let $Y^X = (Y^X, A_{Y^X}, R_{Y^X})$. We define a mapping $e_{X,Y} : (X \times Y^X) \times Y \to H$ as follows:

$$e_{X,Y}((x, f), y) = f(x, y)\ \forall (x, f) \in X \times Y^X,\ \forall y \in Y.$$

Then, by the proofs of Theorem 4.8 in [13] and Theorem 3.15, it can be easily seen that $e_{X,Y} : \mathbf{X}\times\ \mathbf{Y}^X \to \mathbf{Y}$ is a **VFRel(H)**-mapping.

For any $Z = (Z, A_Z, R_Z) \in$ **VFRel(H)**, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be a

VFRel(H)-mapping, where $\mathbf{X} \times \mathbf{Z} = (X \times Z, A_X \times A_Z, R_X \times R_Z)$. We define a mapping $\bar{h} : Z \times (X \times Y) \to H$ as follows: $\forall z \in Z,\ \forall x \in X,\ \forall y \in Y$,

$$\bar{h}(z)(x,y) = h((x,z),y), \text{ where } \bar{h}(z)(x,y) = \bar{h}(z,(x,y)).$$

From the similar arguments of the proofs of Theorem 4.8 in [13] and Theorem 3.15, and the definitions of A_{Y^X} and R_{Y^X}, we can easily prove that $\bar{h} : \mathbf{Z}\times \mathbf{Y}^X$ is a unique **VFRel(H)**-mapping. Such that $e_{X,Y} \circ (I_X \times \bar{h}) = h$. This completes the proof. □

5. The relations between Rel(H) and VRel(H)

Definition 5.1 [11]. The concrete category **Rel(H)** is defined by: Objects are (X, R_X), called an H-*fuzzy relational space*(or simply, a *fuzzy relational sapce*), where X is any set and $R_X \in H^{X\times X}$. A morphism $f : (X, R_X) \to (Y, R_Y)$ is a mapping from X to Y satisfying $R_X(x,y) \leq R_Y(f(x), f(y)),\ \forall(x,y) \in X \times X$, i.e., $R_X \subset f^{-2}(R_Y)$ where " $\leq$ " means the order induced by the operation " $\wedge$ " or " $\vee$ " in H. Every **Rel(H)**-morphism is called a **Rel(H)**-*mapping.*

Lemma 5.2. Define $F : \mathbf{Rel(H)} \to \mathbf{VRel(H)}$ by $F(X, R_X) = (X, R_X)$ and $F(f) = f$. Then F is a functor.

Proof. It is clear that $F(X, R_X) = (X, R_X) \in \mathbf{VRel(H)}$, $\forall(X, R_X) \in \mathbf{Rel(H)}$. Let $(X, R_X), (Y, R_Y) \in \mathbf{Rel(H)}$ and let $f : (X, R_X) \to (Y, R_Y)$ be a **Rel(H)**- mapping. Then $R_X(x, y) \leq R_Y(f(x), f(y)),\ \forall(x, y) \in X \times X$. Since $f : X \to Y$ is a mapping, $f : X \to Y$ is a strong H-fuzzy mapping w.r.t. $I_X \in E_H(X)$ and $I_Y \in E_H(Y)$. Moreover, for each $(x, x') \in X \times X$,

$$\begin{aligned} f^{-2}(R_Y)(x,x') &= \bigvee_{(y,y')\in Y\times Y} [R_Y(y,y') \wedge f(x,y) \wedge f(x',y')] \\ &\geq R_Y(y_\circ, y'_\circ) \\ &\quad [\text{ Since } f \text{ is strong, } \exists y_0 \in Y \text{ and } y'_\circ \in Y \text{ such that} \\ &\quad\ f(x,y_\circ) = 1 \text{ and } f(x',y'_\circ) = 1.] \\ &= R_Y(f(x), f(x')) \\ &\geq R_X(x,x'). \end{aligned}$$

Thus $R_X \subset f^{-2}(R_Y)$. So $F(f) = f \in \mathbf{VRel(H)}$. Hence $F(f) = f : (X, R_X) \to (Y, R_Y)$ is a **VRel(H)**-mapping. Therefore F is a functor. □

Lemma 5.3. We define G : **VRel(H)** $\to$ **Rel(H)** by $G(X, R_X) = (X, R_X)$ and $G(f) = f_*$, where if $f : X \to Y$ is an H-fuzzy mapping w.r.t. $E_X \in E_H(X)$ and $E_Y \in E_H(Y)$, then $f_* : X \times Y \to 2 = \{0,1\}$ is a mapping defined by $f_*(x,y) = f(x,y)$, $\forall (x,y) \in X \times Y$, and E^*_X and E^*_Y are H-fuzzy equalities on X and Y defined by $E^*_X = I_X$ and $E^*_Y = I_Y$, respectively. Then G is a functor.

Proof. It is clear that $G(X, R_X) = (X, R_X) \in$ **Rel(H)**, $\forall (X, R_X) \in$ **VRel(H)**. Let $(X, R_X), (Y, R_Y) \in$ **VRel(H)** and let $f : (X, R_X) \to (Y, R_Y)$ be a **VRel(H)**-mapping. Then $R_X \subset f^{-2}(R_Y)$. By the definition of $G(f)$, $G(f) = f_* : X \to Y$ is a mapping. Let $(x, x') \in X \times X$. Since f is strong, $\exists (y_\circ, y'_\circ) \in Y \times Y$ such that $f(x, y_\circ) = 1 = f(x', y'_\circ)$. Thus

$$\begin{aligned} R_Y(f_*(x), f_*(x')) &= R_Y(f(x), f(y)) \\ &= f^{-2}(R_Y)(x, x') \\ &\quad [\text{ Since } f^{-2}(R_Y)(x,x') = \bigvee_{(y,y')\in Y\times Y} [R_Y(y,y') \\ &\quad \wedge f(x,y) \wedge f(x',y')] = R_Y(f(x), f(x')) \ ] \\ &\geq R_X(x, x'). \end{aligned}$$

So $f_* : (X, R_X) \to (Y, R_Y)$ is a **Rel(H)**-mapping. Hence G is a functor. □

Lemma 5.4. The functor F is a left adjoint of the functor G.

Proof. For each $(X, R_X) \in$ **Rel(H)**, $I_X : (X, R_X) \to GF(X, R_X) = (X, R_X)$ is a **Rel(H)**-mapping. Let $(Y, R_Y) \in$ **VRel(H)** and let $f : (X, R_X) \to G(Y, R_Y)$ be a **Rel(H)**-mapping. Then $R_X(x, x') \leq R_Y(f(x), f(x'))$, $\forall (x, x') \in X \times X$. Thus $R_X \subset f^{-2}(R_Y)$. So $f : F(X, R_X) = (X, R_X) \to (Y, R_Y)$ is a **VRel(H)**-mapping. Hence I_X is a G-universal map for (X, R_X) in **Rel(H)**. This completes the proof. □

Let **VRel$_*$(H)** denote the category with Mor(**VRel$_*$(H)**)= $\{f_* : f \in Mor$ (**VRel(H)**)$\}$. Then clearly **VRel$_*$(H)** is a full subcategory of **VRel(H)**.

Theorem 5.5. Two categories **Rel(H)** and **VRel$_*$(H)** are isomorphic.

Proof. By Lemma 5.2, it is clear that F : **Rel(H)** $\to$**VRel$_*$(H)** is a functor. Also, By Lemma 5.3, G :**VRel$_*$(H)**$\to$ **Rel(H)** is a functor. Let $(X, R_X) \in$ **Rel(H)**. Then clearly $F(X, R_X) = (X, R_X)$. Thus $GF(X, R_X) = (X, R_X)$.

Thus $G \circ F = 1_{\mathbf{Rel(H)}}$. Similary, we can easily see that $F \circ G = 1_{\mathbf{VRel}_*\mathbf{(H)}}$. So F :**Rel(H)**$\rightarrow$ $\mathbf{VRel}_*\mathbf{(H)}$ is an isomorhism. This completes the proof. □

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