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Subcategories of the Category VRel(H)

Kul Hur

Division of Mathematics and Informational Statistics
and Nanoscale Science and Tecchnology Institute
Wonkwang University, Iksan
Chonbuk, Korea 570-749
kulhur@wonkwang.ac.kr

Wang Ro Lee

Faculty of Liberal Education
CheonBuk National University
Jeonju, Cheonbuk, Korea 561-756
wrlee@jbnu.ac.kr

Abstract

We introduce the subcategory $\mathbf{VRel}_R\mathbf{(H)}$ of $\mathbf{VRel(H)}$ consisting of H-fuzzy reflexive relational space on sets and we study structures of $\mathbf{VRel}_R\mathbf{(H)}$ in a viewpoint of the topological universe introduced by L.D.Nel. We show that $\mathbf{VRel}_{PO}\mathbf{(H)}$ of $\mathbf{VRel}_R\mathbf{(H)}$. We show that $\mathbf{VRel}_{PR}\mathbf{(H)}$ [resp. $\mathbf{VRel}_P\mathbf{(H)}$ and $\mathbf{VRel}_E\mathbf{(H)}$] is a topological universe over **Set**.

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1. Introduction

In [6,7], we studied categorical structures of the category $\mathbf{Set(H)}$ consisting of H-fuzzy sets and the category $\mathbf{Rel(H)}$ consisting of H-fuzzy relational spaces in a viewpoint of topological universe, defined by L.D.Nel (cf. [13]). Also, recently, Hur et al.[8,9] made new categories **VSet(H)** and **VRel(H)**, and investigated them in the sense of topological universe, respectively.

In this paper, we study categorical structures of the subcategory $\mathbf{VRel}_R\mathbf{(H)}$ of **VRel(H)** consisting of H-fuzzy reflexive relational spaces on sets in a viewpoint of a topological universe. in particular, it is very interesting that final structures and exponential objects in $\mathbf{VRel}_R\mathbf{(H)}$ are shown to be quite different from those in **VRel(H)** (see [10]). Also, we introduce the subcategories $\mathbf{VRel}_{PR}\mathbf{(H)}$, $\mathbf{VRel}_P\mathbf{(H)}$, $\mathbf{VRel}_E\mathbf{(H)}$, and $\mathbf{VRel}_{PO}\mathbf{(H)}$ of $\mathbf{VRel}_R\mathbf{(H)}$. We investigate some categorical structures of these categories.

For general background for fuzzy set theory, we refer to [11,16,17] and for general categorical background to [4,5,12,13].

2. Preliminaries

We introduce some well-known results [5,12] which are needed in a later section.

Result 2.A [12, Theorem 2.4; 5, Propositions 36.10 and 36.11]. *Let* **A** *be a well-powered and co-(well-powered) topological category. Then the following are equivalent*:

(1) **B** *is epireflective in* **A**.

(2) **B** *is closed under the formation of initial monosources.*

(3) **B** *is closed under the formation of products and pullbacks in* **A**.

Result 2.B [12, Theorem 2.5]. *Let* **A** *be a well-powered and co-(well-powered) topological category. Then the following are equivalent*:

(1) **B** *is bireflective in* **A**.

(2) **B** *is closed under the formation of initial sources.*

Result 2.C [12, Theorem 2.6]. *If* **A** *is a topological category and* **B** *is a bireflective subcategory of* **A**, *then* **B** *is also a topological category. Moreover, every source in* **B** *which is initial in* **A** *is initial in* **B**.

Definition 2.1[8]. A mapping $E_X : X \times X \longrightarrow H$ is called *an H-fuzzy equality on X* if it satisfies the following conditions:

(i) $E_X(x,y) = 1 \;\Leftrightarrow x = y \; \forall x,y \in X$,

(ii) $E_X(x,y) = E_X(y,x) \; \forall x,y \in X$,

(iii) $E_X(x,y) \wedge E(y,z) \leq E_X(x,z) \; \forall x,y,z \in X$.

We will denote the set of all H-fuzzy equalities on X as $E_H(X)$.

Definition 2.2[8]. A H-fuzzy relation f on $X \times Y$ is called an *H-fuzzy mapping w.r.t. $E_X \in \mathrm{E}_H(X)$ and $E_Y \in \mathrm{E}_H(Y)$*, denoted by $f : X \longrightarrow Y$, if it satisfies the following conditions:

(i) $\forall x \in X, \; \exists y \in Y$ such that $f(x,y) > 0$,

(ii) $\forall x,y \in X, \; \forall z,w \in y, \; f(x,z) \wedge f(y,w) \wedge E_X(x,y) \leq E_Y(z,w)$.

Definition 2.3[8]. The *identity H-fuzzy mapping I_X* on X is an H-fuzzy relation on $X \times X$ defined by

$$I_X(x,y) = \begin{cases} 1, & \text{if x=y,} \\ 0, & \text{if x} \neq \text{ y, } \; \forall \text{x,y} \in \text{X} \end{cases}$$

It is clear that $I_X \in \mathrm{E}_H(X)$. Also, if $f : X \rightarrow Y$ is an (ordinary) mapping, then it is an H-fuzzy mapping w.r.t. $I_X \in \mathrm{E}_H(X)$ and $I_Y \in \mathrm{E}_H(Y)$.

Definition 2.4[8]. Let $f : X \rightarrow Y$ be an H-fuzzy mapping w.r.t. $E_X \in \mathrm{E}_H(X)$ and $E_Y \in \mathrm{E}_H(Y)$. Then f is said to be:

(i)*strong* if $\forall x \in X$, $\exists y \in Y$ such that $f(x,y) = 1$,

(ii)*surjective* if $\forall y \in Y$, $\exists x \in X$ such that $f(x,y) > 0$,

(iii)*strong surjective* if $\forall y \in Y$, $\exists x \in X$ such that $f(x,y) = 1$,

(iv)*injective* if $f(x,z) \wedge f(y,w) \wedge E_Y(z,w) \leq E_X(x,y)$, $\forall x,y \in X$, $\forall z,w \in Y$,

(v)*bijective* if it surjective and injective,

(vi)*strong surjective* if it is strong surjective and injective.

In particular, if $f(x,y) = 1$, then we will write $y = f(x)$. It is clear that I_X is a strong H-fuzzy mapping w.r.t. $E_X \in \mathrm{E}_H(X)$. Moreover, it is strong bijective w.r.t. $E_X \in \mathrm{E}_H(X)$.

Definition 2.5[8]. Let $f : X \to Y$ be an H-fuzzy mapping w.r.t. $E_X \in \mathrm{E}_H(X)$ and $E_Y \in \mathrm{E}_H(Y)$, let $A \in H^X$ and let $B \in H^X$.

(i) The *image of A under f*, denoted by $f(A)$, is an H-fuzzy set in Y defined as follows:

$$f(A)(y) = \bigvee_{x \in X} [A(x) \wedge f(x, y)] \quad \forall y \in Y.$$

(ii) The *preimage of B under f*, denoted by $f^{-1}(B)$, is an H-fuzzy set in X defined as follows:

$$f^{-1}(B)(x) = \bigvee_{y \in Y} [B(y) \wedge f(x, y)] \quad \forall x \in X.$$

3. The category $\mathbf{VRel}_R(\mathbf{H})$

Definition 3.1[7]. An H-fuzzy relation R in a set X is said to be *reflexive* if $R(x, x) = 1$ for all $x \in X$.

The class of all H-fuzzy reflexive relational spaces and $\mathbf{VRel(H)}$-mappings between them forms a subcategory of $\mathbf{VRel(H)}$ denoted by $\mathbf{VRel}_R(\mathbf{H})$.

We can easily obtain the following results.

Lemma 3.2. $\mathbf{VRel}_R(\mathbf{H})$ *is properly fibred over* $\mathbf{Set}$.

Lemma 3.3. $\mathbf{VRel}_R(\mathbf{H})$ *is closed under the formation of initial sources in* $\mathbf{VRel(H)}$.

Proof. Let $(f_\alpha : (X, R) \to (X_\alpha, R_\alpha))_\Gamma$ be an initial source in $\mathbf{VRel}_R(\mathbf{H})$ such that each (X_α, R_α) belongs to $\mathbf{Rel}_R(\mathbf{H})$. Take any $x \in X$. Since R_α is reflexive for each $\alpha \in \Gamma$, $R(x, x) = [\bigcap_{\alpha \in \Gamma} f_\alpha^{-2}(R_\alpha)](x, x) = 1$ and hence R is reflexive, where $f_\alpha^{-2} = (f_\alpha \times f_\alpha)^{-1}$. This completes the proof. $\square$

Hence, by Results 2.A and 2.C, we obtain the following result.

Theorem 3.4. (a) *The category* $\mathbf{VRel}_R\mathbf{(H)}$ *is a bireflective subcategory of* $\mathbf{VRel(H)}$.

(b) *The category* $\mathbf{VRel}_R\mathbf{(H)}$ *is topological over* **Set**.

We show that $\mathbf{VRel}_R\mathbf{(H)}$ is cotopological over **Set**, directly.

Theorem 3.5. *The category* $\mathbf{VRel}_R\mathbf{(H)}$ *has final structures over* **Set**.

Proof. Let X be any set and $((X_\alpha, R_\alpha))_\Gamma$ any family of H-fuzzy reflexive relational space indexed by a class Γ. Suppose $(f_\alpha : X_\alpha \to X)_\Gamma$ is a sink of maps. Define $R : X \times X \to H$ by

$$R(x,y) = \begin{cases} [\ \bigcup\limits_{\alpha\in\Gamma} f_\alpha^2(R_\alpha)\ ](x,y), & \text{if (x,y)}\in(\text{X}\times\text{ X}-D_X), \\ 1, & \text{if (x,y)}\in D_X, \end{cases}$$

where $D_X = \{(x,x) \mid x \in X\}$. Then we can easily check that $(f_\alpha : (X_\alpha, R_\alpha) \to (X,R))_\Gamma$ is a final sink in $\mathbf{VRel}_R\mathbf{(H)}$. □

Theorem 3.6. *Final episinks in* $\mathbf{VRel}_R\mathbf{(H)}$ *are preserved by pullbacks.*

Proof. Let $(g_\alpha : (X_\alpha, R_\alpha) \to (Y, R_Y))_\Gamma$ be any final episink in $\mathbf{VRel}_R\mathbf{(H)}$ and $f : (W, R_W) \to (Y, R_Y)$ any $\mathbf{VRel(H)}$-mapping, where (W, R_W) is an H-fuzzy reflexive relational space. For each $\alpha \in \Gamma$, let us take U_α, R_{U_α}, and P_α as in the proof of Lemma 3.13 in [9]. Since $\mathbf{VRel}_R\mathbf{(H)}$ is closed under the formation of pullbacks in $\mathbf{VRel(H)}$ by Result 2.A, it is enough to show that $(e_\alpha)_\Gamma$ is final.

Suppose R^* is the final H-fuzzy relation W with respect to $(e_\alpha)_\Gamma$. Then, for any $(w,w') \in (W \times W - D_W)$,

$$\begin{aligned}
& R_W(w,w') \\
&= R_W(w,w') \wedge R_W(w,w') \\
&\leq R_W(w,w') \wedge f^{-2}(R_Y)(w,w') \\
&= R_W(w,w')(\bigvee_{(y,y')\in Y\times Y} \bigvee_{\alpha\in\ \Gamma} [g_\alpha^2(R_\alpha)(y,y') \wedge f(w,y) \wedge f(w',y')]) \\
&\quad \text{[Since } (g_\alpha)_\Gamma \text{ is final]} \\
&= R_W(w,w') \wedge (\bigvee_{(y,y')\in Y\times Y} \bigvee_{\alpha\in\ \Gamma} \bigvee_{(x_\alpha,y_\alpha)\in X_\alpha\times X_\alpha} [R_\alpha(x_\alpha,y_\alpha) \wedge g_\alpha(x_\alpha,y) \wedge (y_\alpha,y') \\
&\quad \wedge f(w,y) f(w',y')])
\end{aligned}$$

$= \bigvee_{\alpha\in\ \Gamma} \bigvee_{(x_\alpha,y_\alpha)\in X_\alpha\times X_\alpha} [R_W(w,w') \wedge R_{U_\alpha}(x_\alpha,y_\alpha) \wedge (\bigvee_{(y,y')\in Y\times Y} g_\alpha(x_\alpha,y) \wedge$
$g_\alpha(y_\alpha,y')$

$\wedge f(w,y) \wedge f(w',y'))]$

$= \bigvee_{\alpha\in\Gamma} \bigvee_{((w,x_\alpha),(w',y_\alpha))\in U_\alpha\times U_\alpha} [R_{U_\alpha}(w,x_\alpha) \wedge R_{U_\alpha}(w',y_\alpha) \wedge e_\alpha((w,x_\alpha),w) \wedge$
$e_\alpha((w',y_\alpha),$

$w')]$ [Since $R_{U_\alpha} = (R_W \times R_\alpha) \mid_{U_\alpha\times U_\alpha}$, f is strong and g is strong surjective]

$= R^*(w,w')$.

Thus $R_W(w,w') \leq R^*(w,w')$, i.e., $R_W \subset R^*$. On the other hand, by a similar argument as in the proof of Lemma 3.13 in [9], we have $R^* \subset R_W$ on $W\times W - D_W$. Hence $R^*(w,w') = R_W(w,w')$ for any $(w,w') \in (W \times W - D_W)$. Now take $w \in D_W$. Then clearly $R^*(w,w') = 1 = R_W(w,w')$. Therefore $R^* = R_W$. This completes the proof. □

Hence, by Lemma 3.2, Theorems 3.4(2) and 3.6, we obtain the following result.

Theorem 3.7. *The category* **VRel**$_R$**(H)** *is a topological universe over* **Set**. *Hence,* **VRel**$_R$**(H)** *is a concrete quasitopos in the sense of E.J. Dubuc*[3].

Remark 3.8. In [14], Y.Noh obtained exponential objects in **Rel**$_R$**(I)**, where $I = [0,1]$. Now, we show that this construction of an exponential object in **Rel**$_R$**(I)** is applicable to the case of **VRel**$_R$**(H)**.

From Theorem 3.15 in [10], it is clear that **VRel**$_R$**(H)** is Cartesian closed over **Set**. However, we construct an exponetial object in **VRel**$_R$**(H)** by using the way of Noh.

Theorem 3.9. *The category* **VRel**$_R$**(H)** *has an exponential object.*

Proof. For any **X**$= (X, R_X)$ and **Y**$= (Y, R_Y) \in$**VRel**$_R$**(H)**, let Y^X be the set of all strong H-fuzzy mappings from X to Y. We define a mapping R:
$Y^X \times Y^X \to H$ as follows: $\forall f, g \in Y^X$,

$$R(f,g) = \begin{cases} 1, & \text{if } D(f,g)=\varnothing \ , \\ \bigwedge_{(x,x')\in\ D(f,g)} (f\times g)^{-1}(R_Y)(x,x'), & \text{otherwise,} \end{cases}$$

where $D(f,g) = \{(x,x') \in X\times X : R_X(x,x') > (f\times g)^{-1}(R_Y)(x,x')\}$. Then, by the definition of R, R is an H-fuzzy reflexive relation on Y^X. Let $\mathbf{Y}^X = (Y^X, R)$. We define a mapping $e_{X,Y} : (X \times Y^X) \times Y \to H$ by

$$e_{X,Y}((x,f),y) = f(x,y) \quad \forall (x,f) \in X \times Y^X,\ \forall y \in Y.$$

And also we define a mapping $E_{X\times Y^X} : (X\times Y^X)\times(X\times Y^X) \to H$ as follows: For any $(x,f),(x',g) \in X \times Y^X$,

$$E_{X\times Y^X}((x,f),(x',g)) = (\bigwedge_{y\in Y} f(x',y) \wedge \bigwedge_{y'\in Y} g(x,y')) \wedge E_X(x,x') \wedge E'_X(x,x'),$$

where $f : X \to Y$ and $g : X \to Y$ are strong H-fuzzy mappings w.r.t. $E_X \in \mathrm{E}_H(X)$, $E_Y \in \mathrm{E}_H(Y)$, and $E'_X \in \mathrm{E}_H(X)$, $E'_Y \in \mathrm{E}_H(Y)$, respectively. Then, by the process of the proof of Theorem 3.15 in [10], $e_{X,Y} : X \times Y^X \to Y$ is a strong H-fuzzy mapping w.r.t. $E_{X\times Y^X} \in \mathrm{E}_H(X \times Y^X)$ and $E = E_Y \times E'_Y \in \mathrm{E}_H(Y)$. Let $(x,f),(x',g) \in X \times Y^X$. Suppose $D(f,g) \neq \varnothing$. Then

$$\begin{aligned}
& e_{X,Y^{-2}}(R_Y)((x,f),(x',g)) \\
= & \bigvee_{(y,y')\in Y\times Y} [R_Y(y,y') \wedge e_{X,Y}((x,f),y) \wedge e_{X,Y}((x',g),y')] \\
= & \bigvee_{(y,y')\in Y\times Y} [R_Y(y,y') \wedge f(x,y) \wedge g(x',y')] \\
= & \bigvee_{(y,y')\in Y\times Y} (f\times g)^{-1}(R_Y)(x,x') \\
\geq & \ R_X(x,x') \\
= & \ R_X(x,x') \wedge R(f,g) \qquad [\text{ Since } R(f,g)=1\ ] \\
= & \ (R_X \times R)((x,f),(x',g')).
\end{aligned}$$

Thus, $(R_X \times R)((x,f),(x'g)) \leq e_{X,Y}{}^{-2}(R_Y)((x,f),(x',g))$.

Suppose $D(f,g) \neq \varnothing$. Then

$$\begin{aligned}
& (R_X \times R)\ ((x,f),(x',g)) \\
= & \ R_X(x,x') \wedge R(f,g) \\
= & \ R_X(x,x') \wedge [\bigwedge_{(a,b)\in D(f,g)} (f\times g)^{-1}(R_Y)(a,b)] \\
= & \ R_X(x,x') \wedge [\bigwedge_{(a,b)\in D(f,g)} (\bigvee_{(y,y')\in Y\times Y} [R_Y(y,y') \wedge f(a,y) \wedge g(b,y')])] \\
\leq & \ \textstyle\bigvee_{(y,y')\in Y\times Y} [R_Y(y,y') \wedge f(x,y) \wedge g(x',y')] \\
= & \ e_{X,Y}{}^{-2}(R_Y)((x,f),(x',g)).
\end{aligned}$$

Thus $(R_X \times R)((x,f),(x',g)) \leq e_{X,Y}{}^{-2}((x,f),(x',g'))$. So, in either cases, $R_X \times R \subset e_{X,Y}{}^{-2}(R_Y)$. Hence $e_{X,Y} : \mathbf{X} \times \mathbf{Y}^{\mathbf{X}} \to \mathbf{Y}$ is a **VRel(H)**-mapping,

For any $\mathbf{Z} = (Z, R_Z) \in \mathbf{VRel(H)}$, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be a $\mathbf{VRel}_R\mathbf{(H)}$-mapping w.r.t. $E_{X\times Z} = E_X \times E_Z \in \mathrm{E}_H(X \times Z)$ and $E_Y \in \mathrm{E}_H(Y)$. We define a mapping $\overline{h} : Z \times (X \times Y) \to H$ as follows : $\forall z \in Z, \forall x \in X, \forall y \in Y$,

$$\overline{h}(z)(x,y) = h((x,z),y), \text{ where } \overline{h}(z)(x,y) = \overline{h}(z,(x,y)).$$

Let $c \in Z$ and let $a, b \in X$. Then

$$\begin{aligned}
\overline{h}(c)^{-2}(R_Y)(a,b) &= \bigvee_{(y,y')\in Y\times Y} [R_Y(y,y') \wedge \overline{h}(c)(a,y) \wedge \overline{h}(c)(b,y')] \\
&= \bigvee_{(y,y')\in Y\times Y} [R_Y(y,y') \wedge h((a,c),y) \wedge h((b,c),y')] \\
&= h^{-2}(R_Y)((a,c),(b,c)) \\
&\geq R_X(a,b) \wedge R_Z(c,c) \quad [\text{Since } h : \mathbf{X}\times\mathbf{Z} \to \mathbf{Y} \text{ is a } \mathbf{VRel(H)}\text{-mapping}] \\
&= R_X(a,b). \quad [\text{Since } R_Z \text{ is reflexive}]
\end{aligned}$$

Thus $\overline{h}(c) : X \to Y$ is a $\mathbf{VRel}_R\mathbf{(H)}$-mapping. Since h is strong, it is clear that $\overline{h}(z)$ is strong. So $\overline{h}(c) \in \mathbf{Y}^{\mathbf{X}}$. Moreover, $\overline{h} : Z \to Y^X$ is a strong H-fuzzy mapping. Now let $c, c' \in Z$.

Suppose $D(\overline{h}(c), \overline{h}(c')) \neq 0$. Then, by the definition of R,

$$\begin{aligned}
&\overline{h}^{\,-2}(R)(c,c') \\
&= \bigwedge_{(a,b)\in D(\overline{h}(c),\overline{h}(c'))} (\overline{h}(c) \times \overline{h}(c'))^{-1}(R_Y)(a,b) \\
&= \bigwedge_{(a,b)\in D(\overline{h}(c),\overline{h}(c'))} \Big(\bigvee_{(y,y')\in Y\times Y'} [R_Y(y,y') \wedge \overline{h}(c)(a,y) \wedge \overline{h}(c')(b,y')]\Big) \\
&= \bigwedge_{(a,b)\in D(\overline{h}(c),\overline{h}(c'))} \Big(\bigvee_{(y,y')\in Y\times Y'} [R_Y(y,y') \wedge h((a,c),y) \wedge h((b,c'),y')]\Big) \\
&= \bigwedge_{(a,b)\in D(\overline{h}(c),\overline{h}(c'))} h^{-2}(R_Y)((a,c),(b,c')) \\
&= \bigwedge_{(a,b)\in D(\overline{h}(c),\overline{h}(c'))} (R_X \times R_Z)((a,c),(b,c')) \\
&\quad [\text{Since } h : X \times Z \to Y \text{ is a } \mathbf{VRel(H)}\text{-mapping}] \\
&= \bigwedge_{(a,b)\in D(\overline{h}(c),\overline{h}(c'))} [R_X(a,b) \wedge R_Z(c,c')].
\end{aligned}$$

On the other hand, let $(a,b) \in D(\overline{h}(c), \overline{h}(c'))$. Then

$$\begin{aligned}
R_X(a,b) &> D(\overline{h}(c) \times \overline{h}(c'))^{-1}(R_Y)(a,b) \\
&= \bigvee_{(y,y')\in Y\times Y} [R_Y(y,y') \wedge \overline{h}(c)(a,y) \wedge \overline{h}(c')(b,y)]
\end{aligned}$$

$$= h^{-2}(R_Y)((a,c),(b,c'))$$
$$\geq R_X(a,b) \wedge R_Z(c,c')$$
[Since $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ is a **VRel(H)**-mapping]

Thus $R_X(a,b) > R_Z(c,c')$. So $\overline{h}^{-2}(R)(c,c') \geq R_Z(c,c')$. In either cases, $R_Z \subset \overline{h}^{-2}(R)$. Hence $\overline{h} : \mathbf{Z} \to \mathbf{Y}^{\mathbf{X}}$ is a $\mathbf{VRel}_R\mathbf{(H)}$-mapping. Moreover, $\overline{h}$ is unique and $e_{X,Y} \circ (I_X \times \overline{h}) = h$. This completes the proof. □

Remark 3.10. (a) We can see that the exponential objects in $\mathbf{VRel}_R\mathbf{(H)}$ are quite different from those **VRel(H)** constructed in Theorem 3.15 in [10].

(b) The category $\mathbf{VRel}_R\mathbf{(H)}$ has no subject classifier.

Example 3.10. Let $H = \{0,1\}$ be the two points chain and $X = \{a,b\}$. Let R_1 and R_2 be the H-fuzzy reflexive relations on X given by

$$\mu_{R_1}(a,a) = \mu_{R_1}(b,b) = 1,\ \mu_{R_1}(a,b) = \mu_{R_1}(b,a) = 0,$$
$$\mu_{R_2}(a,a) = \mu_{R_2}(b,b) = 1,\ \mu_{R_2}(a,b) = \mu_{R_2}(b,a) = 0.$$

Let $I_X : (X,R_1) \to (X,R_2)$ be the identity H-fuzzy mapping. Then clearly, I_X is both monomorphism and epimorphism in $\mathbf{VRel}_R(H)$. However, I_X is not an isomorphism in $\mathbf{VRel}_R(H)$. Hence $\mathbf{VRel}_R(H)$ has no subobject classifier(cf.[2]). □

4. Subcategories of $\mathbf{VRel}_R(H)$

We introduce some subcategories of $\mathbf{VRel}_R(H)$ which are topological universes over **Set**.

Definition 4.1[7]. Let R be an H-fuzzy relation on a set X.

(1) R is *symmetric* if and only if $R(x,y) = R(y,x)$ for all $x,y \in X$.

(2) R is *perfectly antisymmetric* if and only if $(x,y) \in x \times X$ with $x \neq y$, if $R(x,y) > 0$ then $R(y,x) = 0$.

(3) R is *(sup-min) transitive* if and only if $R \circ R \subset R$, where

$$R \circ R(x,y) = \bigvee_{z \in X} [R(x,z) \wedge R(z,y)] \text{ for all } (x,y) \in X \times X.$$

(4) R is an *H-fuzzy proximity relation* if and only if R is reflexive and symmetric.

(5) R is an *H-fuzzy preorder relation* if and only if R is reflexive and transitive.

(6) R is an *H-fuzzy similarity relation* if and only if R is reflexive and symmetric and transitive.

(7) R is an *H-fuzzy perfect order relation* if and only if R is reflexive, transitive and perfectly antisymmetirc.

Notation (1) $\mathbf{VRel}_S(H)$ denotes the full subcategory of $\mathbf{VRel}(H)$ determined by all H-fuzzy symmetric relational spaces.

(2) $\mathbf{VRel}_T(H)$ denotes the full subcategory of $\mathbf{VRel}(H)$ determined by all H-fuzzy max-min transitive relational spaces.

(3) $\mathbf{VRel}_{PR}(H)$=$\mathbf{VRel}_R(H)\cap\mathbf{VRel}_S(H)$ denotes the full subcategory of $\mathbf{VRel}_R(H)$ determined by all objects (X, R), where R is an H-fuzzy proximity relation on X.

(4) $\mathbf{VRel}_P(H)$=$\mathbf{VRel}_R(H)\cap\mathbf{VRel}_T(H)$ denotes the full subcategory of $\mathbf{VRel}_R(H)$ determined by all objects (X, R), where R is an H-fuzzy preorder relation on X.

(5) $\mathbf{VRel}_E(H)$=$\mathbf{VRel}_R(H)\cap\mathbf{VRel}_S(H)\cap$ $\mathbf{VRel}_T(H)$denotes the full subcategory of $\mathbf{VRel}_R(H)$ determined by all objects (X, R), where R is an H-fuzzy perfect order relation on X.

(6) $\mathbf{VRel}_{PO}(H)$ denotes the full subcategory of $\mathbf{VRel}_R(H)$ determined by all objects (X, R), where R is an H-fuzzy perfect order relation on X.

It is easy to show that the following results hold.

Lemma 4.2. The category $\mathbf{VRel}_{PR}(H)$[resp. $\mathbf{VRel}_P(H)$ and $\mathbf{VRel}_E(H)$, and $\mathbf{VRel}_{PO}(H)$] is properly fibred over **Set**.

Lemma 4.3. The category $\mathbf{VRel}_{PR}(H)$[resp. $\mathbf{VRel}_P(H)$, and $\mathbf{VRel}_E(H)$] is closed under the formation of initial sources in $\mathbf{VRel}_R(H)$.

Proof. Let $(f_\alpha : (X, R) \to (X_\alpha, R_\alpha))_\Gamma$ be an initial source in $\mathbf{VRel}_R(H)$ such that each (X_α, R_α) belongs to $\mathbf{VRel}_{PR}(H)$[resp. $\mathbf{VRel}_P(H)$] and $\mathbf{VRel}_E(H)$. Then clearly, by the definition of R, R is reflexive and symmetric and thus it is enough to show that R is transitive. Take any $x, y \in X$. Then

$$
\begin{aligned}
&(R \circ R)(x,y)\\
&= \bigvee_{z\in X}[R(x,z) \wedge R(z,y)]\\
&= \bigvee_{z\in X}[\bigcap_{\alpha\in\Gamma} f_\alpha^{-2}(R_\alpha)(x,z) \wedge \bigcap_{\alpha\in\Gamma} f_\alpha^{-2}(R_\alpha)(z,y)] \text{ [Since } R \text{ is an initial structure]}\\
&= \bigvee_{z\in X}\{\bigwedge_{\alpha\in\Gamma}(\bigvee_{(a,b)\in X_\alpha\times X_\alpha}[R_\alpha(a,b) \wedge f_\alpha(x,a) \wedge f_\alpha(z,b)]) \wedge \bigwedge_{\alpha\in\Gamma}(\bigvee_{(b,c)\in X_\alpha\times X_\alpha}\\
&\qquad [R_\alpha(b,c) \wedge f_\alpha(z,b) \wedge f_\alpha(y,c)])\}\\
&\le \bigwedge_{\alpha\in\Gamma}\bigvee_{(a,c)\in X_\alpha\times X_\alpha}[(R_\alpha \circ (R_\alpha)(a,c) \wedge f_\alpha(x,a) \wedge f_\alpha(y,c)]\\
&\le \bigwedge_{\alpha\in\Gamma}\bigvee_{(a,c)\in X_\alpha\times X_\alpha}[(R_\alpha(a,c) \wedge f_\alpha(x,a) \wedge f_\alpha(y,c)] \text{ [Since } R_\alpha \text{ is transitive]}\\
&= \bigwedge_{\alpha\in\Gamma} f_\alpha^{-2}(R_\alpha)(x,y)\\
&= [\bigcap_{\alpha\in\Gamma} f_\alpha^{-2}(R_\alpha)](x,y)\\
&= R(x,y).
\end{aligned}
$$

This completes the proof. □

Hence, by Result 2.B and Lemma 4.3, we obtain the following result.

Theorem 4.4. $\mathbf{VRel}_{PR}(H)$, $\mathbf{VRel}_{P}(H)$, and $\mathbf{VRel}_{E}(H)$ are bireflexive subcategories of $\mathbf{VRel}_{R}(H)$ and hence topological categories over **Set**.

Theorem 4.5. $\mathbf{VRel}_{PR}(H)$, $\mathbf{VRel}_{P}(H)$, and $\mathbf{VRel}_{E}(H)$ are closed under the formation of final structures in $\mathbf{VRel}_{R}(H)$ and hence all of them are bicoreflective subcategories of $\mathbf{VRel}_{R}(H)$.

Proof. Let $(f_\alpha : (X_\alpha, R_\alpha) \to (X,R))_\Gamma$ be any final sink in $\mathbf{VRel}_{R}(H)$ such that each (X_α, R_α) belongs to $\mathbf{VRel}_{PR}(H)$ (resp. $\mathbf{VRel}_{P}(H)$, and $\mathbf{VRel}_{E}(H)$). Then clearly, by the definition of R, R is reflexive and symmetric and thus it is enough to show that R is transitive. Take any $x, y \in X$. Then, from the proof of Theorem 3.5,

$$
\begin{aligned}
&(R \circ R)(x,y)\\
&= \bigvee_{z\in X}[R(x,z) \wedge R(z,y)]
\end{aligned}
$$

$$= \bigvee_{z\in X} \{[\bigcup_{\alpha\in\Gamma} f_\alpha(R_\alpha)](x,z) \wedge [\bigcup_{\alpha\in\Gamma} f_\alpha(R_\alpha)(z,y)]\} \text{ [Since } R \text{ is a final structure]}$$
$$= \bigvee_{z\in X} \{\bigvee_{\alpha\in\Gamma}(\bigvee_{(a,b)\in X_\alpha\times X_\alpha} [R_\alpha(a,b) \wedge f_\alpha(a,x) \wedge f_\alpha(b,z)]) \wedge \bigvee_{\alpha\in\Gamma}(\bigvee_{(b,c)\in X_\alpha\times X_\alpha} [R_\alpha(b,c) \wedge f_\alpha(b,z) \wedge f_\alpha(c,y)])\}$$
$$\leq \bigvee_{\alpha\in\Gamma}(\bigvee_{(a,c)\in X_\alpha\times X_\alpha} [(R_\alpha \circ (R_\alpha)(a,c) \wedge f_\alpha(a,x) \wedge f_\alpha(c,y)])$$
$$\leq \bigvee_{\alpha\in\Gamma}(\bigvee_{(a,c)\in X_\alpha\times X_\alpha} [(R_\alpha(a,c) \wedge f_\alpha(a,x) \wedge f_\alpha(c,y)]) \text{ [Since } R_\alpha \text{ is transitive]}$$
$$= \bigvee_{\alpha\in\Gamma} f_\alpha(R_\alpha)(x,y)$$
$$= [\bigcup_{\alpha\in\Gamma} f_\alpha(R_\alpha)](x,y)$$
$$= R(x,y).$$

Hence R is an H-fuzzy transitive relation on X. This completes the proof. □

Theorem 4.6. Final episinks in $\mathbf{VRel}_{PR}(H)$ [resp. $\mathbf{VRel}_P(H)$ and $\mathbf{VRel}_E(H)$] are preserved by pullbacks.

Proof. It is obvious, by Lemma 3.13 in [10] and Theorem 4.5, and from the fact that $\mathbf{VRel}(H)$, by Result 2.A.

Hence, by Theorem 4.4 and 4.6, we obtain the following result.

Theorem 4.7. *The category* $\mathbf{VRel}_{PR}\mathbf{(H)}$ [*resp.* $\mathbf{VRel}_P\mathbf{(H)}$ *and* $\mathbf{VRel}_E\mathbf{(H)}$] *is a topological universe over* **Set**. *Hence, each category is a concrete quasitopos in the sense of E.J. Dubuc* [3].

Theorem 4.8. *The* $\mathbf{VRel}_{PO}\mathbf{(H)}$ *is closed under the formation of initial monosource in* $\mathbf{VRel}_R\mathbf{(H)}$ *and hence it is an epireflective subcategory of* $\mathbf{VRel}_R\mathbf{(H)}$ *by Result 2.A.*

Proof. Let $(f : (X,R) \to (X_\alpha, R_\alpha))_\Gamma$ be initial monosource in $\mathbf{VRel}_R\mathbf{(H)}$, where $(X_\alpha, R_\alpha) \in \mathbf{Ob(VRel}_{PO}\mathbf{(H))}$ for each $\alpha \in \Gamma$. In Lemma 4.3, reflexivity and transitivity are proved. For any $(x,y) \in X \times X$ with $x \neq y$, suppose $R(x,y) > 0$. Then $[\bigcap_{\alpha\in\Gamma} f_\alpha^{-2}(R_\alpha)](x,y) > 0$. Since $(f_\alpha : (X,R) \to (X_\alpha, R_\alpha))_\Gamma$ is a monosource, there exists a set $\varnothing \neq \Gamma' \subset \Gamma$ such that $f_\alpha(x,y) \neq f_\beta(x,y)$ for all

$\beta \in \Gamma'$. Thus, for each $\beta \in \Gamma'$, $[f_\beta^{-2}(R_\beta)](y,x) = 0$. So $[\bigcap_{\alpha\in\Gamma} f_\alpha^{-2}(R_\alpha)](y,x) = 0$. Hence R is perfectly antisymmetric. This completes the proof. □

References

[1] G. Birkhoff, *Lattice Theory*(A.M.S. Colloquium Publication, Vol. XXV, Providence, RI, 1967).

[2] J.C. Carrega, The category Set(H) and Fuz(H), *Fuzzy Sets and Systems*, 9(1983), 327-332.

[3] E.J. Dubuc, Concrete quasitopol, applications of sheaves, *Proc. Dunham 1977, Lect. Notes in Math.* 753(1976), 239-254.

[4] H. Herrlich, Cartesian closed topological categories, *Math. Coll. Univ. Cape Town*, 9(1974), 1-16.

[5] H. Herrlich and G.E. Streker, *Category Theory* (Allyn and Bacon, Newton, MA, 1973).

[6] K. Hur, H-fuzzy relations(I): A topological universe viewpoint, *Fuzzy Sets and Systems*, 61(1995), 239-244.

[7] K.Hur, H-fuzzy relations(II), A topological universe viewpoint, Fuzzy Sets and Systems, 63(1994), 73-79.

[8] K. Hur, A note on the category Set(H) , *Honam Math. J.* 10(1988), 89-94.

[9] K. Hur, S.R.Kim and P.K.Lim, The category VSet(H), To submit.

[10] K. Hur, S.H.Kim and P.K.Lim, The category VRel(H), To submit.

[11] A. Kaufmann, *Introduction to the Theory of Fuzzy Subsets,* Vol. 1(Academic Press, New York, 1975).

[12] C.Y.Kim, S.S.Hong, Y.H.Hong and P.H.Park, *Algebras in Cartesian closed Topological Categories,* Lecture Note Series 1, Seminars in Math. Yon Sei University (1979).

[13] L.D.Nel, Topological universes and smooth Gelfand-Naimark, duality, Mathematical applications of category theory, Proc. A.M.S. Spec. Session Denver, 1983, *Contemporary Mathematics*, 30(1984), 224-276.

[14] Y.Noh, Categorical aspects of fuzzy relations, M.A. Thesis, Yon Sei University.

[15] D. Ponasse, Some remarks on the category Fuz(H) of M. Eytan, *Fuzzy Sets and Systems*, 9(1983), 199-204.

[16] L.A.Zedeh, Fuzzy sets, In form and Control, 8(1965), 338-353.
[17] ____, Similarity relations and fuzzy orderings, *Inform.* 3(1971), 177-200.

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