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Optimal Control of Multi-Item Inventory Model with Natural Deterioration Function

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Abstract

The paper provides an extension of the optimal control problem of two-item inventory model with deteriorating items. The total cost includes the sum of the holding costs of inventory, the holding costs of one item due to the presence of the other and the production costs and the profit out due to presence of two types of deterioration. The solution of optimal control problem of two-item model will be carried out using Pontrygin principle. The controlled system of non-linear differential equations will be solved numerically using four different types of demand rates and two types of the natural deterioration rate. This paper dealing with the deterioration rates as a functions of the inventory levels and the time.

Keywords: Multi-item inventory, Deteriorating items, Pontryagin principle, Natural deterioration rates

1 Introduction

The literature on multi-item dynamic inventory models is really sparse, since most of the classical studies are concerned with a single-item inventory model. We cite some of the most recent ones in order to give an idea on the wide range of optimal control applications in the multi-item inventory-production system. Ben-Daya and Raouf [1] have developed approach for a more realistic and general SPIP (Single Period Inventory Problem), they consider a multi-item with budgetary and floor- or shelf- space constraints, they assume that, the demand of the items follows uniform probability distribution. Also, they discussed a multi-item inventory model with stochastic demand subject to the restrictions on available space and budget. El-Gohary and El-sayed presented the optimal control of multi-item inventory model with deteriorating items for different types of demand

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rates and fixed natural deterioration rates [3]. Foul, El-Gohary and El-sayed [4] have discussed the optimal control of HMMS reverse logistics model with deteriorating items. Bhattacharya [2] has studied a two-item inventory for deteriorating items with a linear stock-dependent demand rates. Lenard and Roy [6] defined another approach for the determination of inventory policies based on the notion of efficient policy surface and extend this notion to multi-item inventory control by defining the concepts of family and aggregate item. Different mathematicians like Worell, Hall and others [11] have applied different programming methods to solve multi-items inventory problems. As in the case of a single-item inventory, Kar and others [5] have also considered, density-dependent demand rate for multi-item inventory. Sulem [10] has been determined the optimal ordering policy for impulse control of a deterministic two product inventory system subject to constant demand rates, linear storage and shortage costs and economies of joint ordering. Rosenblatt [8] has discussed multi-item inventory system with budgetary constraint comparison between the lagrangian and the fixed cycle approach.
The present paper is organized as follows:
The mathematical model of two-item inventory model and the mathematical formulation of the optimal control problem of this model. Also, the optimal production rates and the optimal inventory levels are derived in this section. The critical economic conditions impose on the production rates are discussed. The numerical solution of the controlled system using different types of the demand rates and natural deteriorating rates. Finally, conclusions of the results will present.

2 Two-item Inventory Model

In this section, we will be concerned with mathematical formulation and optimal control of two-item inventory model for deteriorating items.
We consider a factory producing two items and having a finished goods warehouse. The objective function includes the sum of the holding costs of inventory , the holding costs of one item due to the presence of other, the production costs and the profit out of presence of two types of the deterioration. The problem is represented as an optimal control problem with two state variables and two control variables which are the inventory levels and production rates respectively. Since the analytical solution of the controlled system is very difficult because of this system is non-linear , we will solve it numerically. The solution of the controlled system includes four different cases of the demand which are: constants, linear functions of inventory levels, logistic functions of inventory levels and periodic functions of time. Also, two types of the natural deterioration rates which are: linear functions of inventory levels and periodic functions of the time.

2.1 The Model Assumptions

This model assumes that the production rates are itself the rates of continuous supply to inventory levels. The inventory goal levels of two items and their goal production rates are determined before by the administration.

Let us define the following parameters:

$x_i(t)$	:	Inventory levels at time t,
$u_i(t)$	:	Production rates at time t,
T	:	Length of the planning period,
$\hat{x}_i$	:	Inventory goal levels,
$\hat{u}_i$	:	Production goal rates,
x_{io}	:	Initial inventory levels,
c_{ii}	:	Production cost coefficients,
h_{ii}	:	Inventory holding cost coefficients,
h_{12}	:	Inventory holding cost coefficient of x_1 due to presence of unit of x_2 or vice-versa,
$D_i(x_1, x_2, t)$	:	Demand rates at the instantaneous level of inventory (x_1, x_2),
a_{ii}	:	Deterioration coefficient due to self-contact of x_i,
a_{ij} $(i \neq j)$	:	the demand coefficient of x_i due to presence of unit of x_j,
$\Theta_i(x_i, t)$	:	Natural deterioration rates of x_i.
p_i	:	Unit price of x_i.

Next we will use the above assumptions and Pontryagin principle to describe the problem of optimal control of a continuous-time two-item inventory model with deteriorating items.

2.2 Optimal Control Problem

The optimal control problem is defined to be admissible production rates which minimize the total cost which given by:

$$\begin{aligned} 2J \;=\; & \min_{u_i(t)\geq 0} \int_0^T \Big\{ \sum_{i=1}^{2} \Big(h_{ii}(x_i - \hat{x}_i)^2 + c_{ii}(u_i - \hat{u}_i)^2 + (\theta_i + a_{ii})x_i^2 \Big) + \\ & 2h_{12}(x_1 - \hat{x}_1)(x_2 - \hat{x}_2) \Big\} dt, \end{aligned} \tag{2.1}$$

subject to

$$\begin{aligned} \dot{x}_1 &= -x_1(t)\Big(\Theta_1 + a_{12}x_2(t) + a_{11}x_1(t)\Big) - D_1(x_1, x_2, t) + u_1(t), \\ \dot{x}_2 &= -x_2(t)\Big(\Theta_2 + a_{21}x_1(t) + a_{22}x_2(t)\Big) - D_2(x_1, x_2, t) + u_2(t), \end{aligned} \tag{2.2}$$

and

$$x_i(t) \geq 0, \;\; u_i(t) \geq 0, \tag{2.3}$$

where $t \in [0, T]$, $h_{11}h_{22} > h_{12}^2$, $h_{ii} > 0$, $c_{ii} > 0$, $i = 1, 2$

The economic interpretation of the objective function (2.1) is that we want to keep the inventory levels (x_1, x_2) as close as possible to its goal levels $(\hat{x}_1, \hat{x}_2)$ and also keep the production rates (u_1, u_2) as close as possible to its goal rates $(\hat{u}_1, \hat{u}_2)$. Also, we want to reduce the lost profits due the presence two types of the deterioration.
Now we replace the cost integral by introducing an additional state variable which satisfies the state equation

$$\begin{aligned} 2\dot{x}_0(t) &= h_{11}(x_1-\hat{x}_1)^2 + c_{11}(u_1-\hat{u}_1)^2 + h_{22}(x_2-\hat{x}_2)^2 + c_{22}(u_2-\hat{u}_2)^2 + \\ & \quad 2h_{12}(x_1-\hat{x}_1)(x_2-\hat{x}_2) + (\theta_1+a_{11})x_1^2 + (\theta_2+a_{22})x_2^2, \end{aligned} \tag{2.4}$$

with initial value and boundary condition $x_0(0) = 0, \;\; x_0(T) = J$ respectively.
Now, we introduce the co-state variables λ_0 , λ_1 and λ_2 corresponding to the state variables x_0 , x_1 and x_2 respectively.
From equations (2.2) and (2.4) we can write the Hamiltonian function

$$H = \lambda_0\dot{x}_0 + \lambda_1\dot{x}_1 + \lambda_2\dot{x}_2, \tag{2.5}$$

In addition to get the co-state equation and the Lagrange multipliers associated with the constraints (2.3) we form the Lagrange function

$$L = H + \mu_1(t)x_1 + \mu_2(t)x_2 + \mu_3(t)u_1 + \mu_4(t)u_2, \tag{2.6}$$

where $\mu_1(t), \mu_2(t), \mu_3(t)$ and $\mu_4(t)$ are called Lagrange multipliers, these Lagrange multipliers satisfy the complementary slackness conditions

$$\begin{gathered} \mu_1(t) \geq 0,\; \mu_2(t) \geq 0,\; \mu_3(t) \geq 0,\; \mu_4(t) \geq 0, \\ \mu_1 x_1(t) = 0,\; \mu_2 x_2(t) = 0,\; \mu_3 u_1(t) = 0,\; \mu_4 u_2(t) = 0, \end{gathered} \tag{2.7}$$

From (2.6) we can easily obtained the co-state equations

$$\dot{\lambda}_0(t) = -\frac{\partial L}{\partial x_0} = 0, \qquad \dot{\lambda}_1(t) = -\frac{\partial L}{\partial x_1}, \qquad \dot{\lambda}_2(t) = -\frac{\partial L}{\partial x_2}, \tag{2.8}$$

The first equation of the system (2.8) shows that the co-state variable $\lambda_0(t)$ is constant and the Pontryagin maximum principle requires that this constant should be negative, without loss of generality [4].
We can choose

$$\lambda_0(t) = -1, \tag{2.9}$$

Substituting from (2.4), (2.5), (2.8) and (2.9) in (2.6) we can write L in the form

$$\begin{aligned} L &= -\frac{1}{2}\Big[h_{11}(x_1-\hat{x}_1)^2 + c_{11}(u_1-\hat{u}_1)^2 + h_{22}(x_2-\hat{x}_2)^2 \\ & \quad + c_{22}(u_2-\hat{u}_2)^2 + 2h_{12}(x_1-\hat{x}_1)(x_2-\hat{x}_2) + \\ & \quad + (\theta_1+a_{11})x_1^2 + (\theta_2+a_{22})x_2^2\Big] \end{aligned} \tag{2.10}$$

$$+\lambda_1\Big[-x_1(\Theta_1+a_{12}x_2+a_{11}x_1)-D_1+u_1\Big]$$
$$+\lambda_2\Big[-x_2(\Theta_2+a_{21}x_1+a_{22}x_2)-D_2+u_2\Big] \tag{2.11}$$
$$+\mu_1x_1+\mu_2x_2+\mu_3u_1+\mu_4u_2,$$

From conditions (2.3) and (2.7) we get

$$\mu_1(t)=\mu_2(t)=\mu_3(t)=\mu_4(t)=0 \tag{2.12}$$

Substituting from (2.10) and (2.12) into (2.8) we get

$$\dot{\lambda}_1=\lambda_1\Big(\frac{\partial D_1}{\partial x_1}+\frac{\partial \Theta_1}{\partial x_1}+a_{12}x_2+2a_{11}x_1\Big)+\lambda_2(\frac{\partial D_2}{\partial x_1}+a_{21}x_2)+h_{11}(x_1-\hat{x}_1)$$
$$+h_{12}(x_2-\hat{x}_2)+p_1(a_{11}+\theta_1)x_1,$$
$$\dot{\lambda}_2=\lambda_2\Big(\frac{\partial D_2}{\partial x_2}+\frac{\partial \Theta_2}{\partial x_2}+a_{21}x_1+2a_{22}x_2\Big)+\lambda_1(\frac{\partial D_1}{\partial x_2}+a_{12}x_1)+h_{22}(x_2-\hat{x}_2) \tag{2.13}$$
$$+h_{12}(x_1-\hat{x}_1)+p_2(a_{22}+\theta_2)x_2,$$

with boundary conditions $\lambda_i(T)=0, \quad i=1,2$

To obtain the control variables $u_i(t)$ we will be differentiate the Lagrange function (2.10) with respect to $u_i(i=1,2)$ and putting

$$\frac{\partial L}{\partial u_i}=0, \quad i=1,2 \tag{2.14}$$

we get

$$u_i(t)=\hat{u}_i+\frac{\lambda_i(t)}{c_{ii}}, \quad u_i(t)\in\Omega_i(t),\ t\in[0,T], \quad i=1,2 \tag{2.15}$$

where $\Omega_i(t)=[0,\ u_{i_{\max(t)}}]$, $u_{i_{\max(t)}}(i=1,2)$ are maximum possible production rates, the sets of all possible production rates which are determined by physical or economic constraints on the value of control variables at time t.

From (2.15) and using boundary conditions $\lambda_i(T)=0(i=1,2)$ we can get

$$u_i(T)=\hat{u}_i, \quad i=1,2 \tag{2.16}$$

To be sure from that $u_i(t)\geq 0$, then must be $\hat{u}_i\geq -\frac{\lambda_i(t)}{c_{ii}}$, i=1,2

From (2.3) and (2.15) we can find the optimal production rates

$$u_i^*(t) = \max\Big[0,\ \hat{u}_i + \frac{\lambda_i(t)}{c_{ii}}\Big], \quad i = 1, 2 \tag{2.17}$$

we can put (2.17) in the form

$$u_i^*(t) = sat_i\Big[0,\ u_{i_{\max(t)}},\ \hat{u}_i + \frac{\lambda_i(t)}{c_{ii}}\Big] = \begin{cases} 0 & \hat{u}_i + \frac{\lambda_i(t)}{c_{ii}} < 0 \\ \hat{u}_i + \frac{\lambda_i(t)}{c_{ii}} & 0 \le \hat{u}_i + \frac{\lambda_i(t)}{c_{ii}} \le u_{i_{\max(t)}} \\ u_{i_{\max(t)}} & \hat{u}_i + \frac{\lambda_i(t)}{c_{ii}} > u_{i_{\max(t)}} \end{cases} \tag{2.18}$$

The sat functions are used to specify the optimal control for linear-quadratic problems and the word "sat" is short for the word "saturation" [11].

Economically, we suppose that $\hat{u}_i \ge x_{i0} > 0\,(i = 1, 2)$. So, the production goal rates $\hat{u}_i$ must be large enough and the initial values x_{i0} small enough so that (2.17) always give nonnegative production values. We can confirm that, in the numerical solution, by choosing small values for $\hat{u}_i$ less than x_{i0}, and then $u_i(t)$ become negative.

From (2.2), (2.13) and (2.15) we can get the following system of non-linear ordinary differential equations:

$$\left.\begin{aligned} &\dot{x}_1 = -x_1\Big(\Theta_1 + a_{12}x_2 + a_{11}x_1\Big) - D_1(x_1, x_2, t) + \hat{u}_1 + \tfrac{\lambda_1}{c_{11}} \\ &\dot{x}_2 = -x_2\Big(\Theta_2 + a_{21}x_1 + a_{22}x_2\Big) - D_2(x_1, x_2, t) + \hat{u}_2 + \tfrac{\lambda_2}{c_{22}} \\ &\dot{\lambda}_1 = \lambda_1\Big(\frac{\partial D_1}{\partial x_1} + \frac{\partial \Theta_1}{\partial x_1} + a_{12}x_2 + 2a_{11}x_1\Big) + \lambda_2(\frac{\partial D_2}{\partial x_1} + a_{21}x_2) + h_{11}(x_1 - \hat{x}_1) \\ &+h_{12}(x_2 - \hat{x}_2) + p_1(a_{11} + \theta_1)x_1 \\ &\dot{\lambda}_2 = \lambda_2\Big(\frac{\partial D_2}{\partial x_2} + \frac{\partial \Theta_2}{\partial x_2} + a_{21}x_1 + 2a_{22}x_2\Big) + \lambda_1(\frac{\partial D_1}{\partial x_2} + a_{12}x_1) + h_{22}(x_2 - \hat{x}_2) \\ &+h_{12}(x_1 - \hat{x}_1) + p_2(a_{22} + \theta_2)x_2 \end{aligned}\right\}, \tag{2.19}$$

This system will be used to describe the time evolution of inventory levels and production rates. The analytical solution of this system is very difficult since this system is non-linear and we will solve it numerically.

3 Numerical Solution

The numerical solution seems necessary in the absence of the analytically solution of the non-linear system (2.19). The solution is based on the numerical integration of this system using Runge-Kutta method. This section displays graphically the numerical integration of this system for different values of monetary and non-monetary parameters. The numerical solution of the system (2.19) will be discussed for four different types of the demand rates which are :

1. Constants: $[D_i = \alpha_i]$,
2. Linear functions of inventory levels: $[D_i = d_i x_i + \alpha_i]$,
3. Logistic functions of inventory levels: $[D_i = x_i(g_i - x_i)]$,
4. Periodic function of time: $[D_i = 1 + k_i \sin t]$.

where α_i , d_i , g_i and k_i $(i = 1, 2)$ are positive constants.
Also, in every demand rate we will use two types of natural deterioration rates:

- $\Theta_i = \theta_i x_i$.
- $\Theta_i = \theta_i sin(t)$.

Table (1) presents the values of system parameters and initial states as follows:

parameter	h_{11}	c_{11}	α_1	θ_1	a_{11}	a_{12}	$\hat{u}_1$	x_{10}	$\hat{x}_1$	h_{12}	d_1	g_1	k_1
value	4	6	0.6	0.02	0.04	0.7	9	2	4	-4	3	10	2
parameter	h_{22}	c_{22}	α_2	θ_2	a_{22}	a_{21}	$\hat{u}_2$	x_{20}	$\hat{x}_2$	T	d_2	g_2	k_2
value	5	5	0.8	0.03	0.05	0.6	8	1	3	5	4	20	1

Table (1)

3.1 Constant demand rates

In this subsection, we will present the model with constant demand rates. Substituting in the controlled system (2.19) by $D_i(x_1, x_2, t) = \alpha_i$, $\Theta_i = \theta_i x_i$ and $\Theta_i = \theta_i sin(t)$, we get the results in table (2) as following:

Natural deterioration rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J(x_i^*(T), u_i^*(T))$
$\Theta_i = \theta_i x_i$	3.76	2.88	9	8	6.15
$\Theta_i = \theta_i sin(t)$	3.83	2.96	9	8	6.30

Table (2)

3.2 Linear demand rates

In this subsection, we will present the model with linear demand rates. Substituting in the controlled system (2.19) by $D_i(x_1, x_2, t) = d_i x_i + \alpha_i$, $\Theta_i = \theta_i x_i$ and $\Theta_i = \theta_i sin(t)$, we get the results in table (3) as following:

Natural deterioration rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J(x_i^*(T), u_i^*(T))$
$\Theta_i = \theta_i x_i$	2.07	1.35	9	8	9.30
$\Theta_i = \theta_i sin(t)$	2.1	1.36	9	8	9.22

Table (3)

Comparing the linear demand rates by the constant demand rates, the inventory levels are decreases when the demand rates are linear and the total costs increase. The difference between them are existed, where the constant demand rates are special cases of the linear demand rates when $d_i = 0 (i = 1, 2)$. But the main difference between them is that: in the constant demand rates some values of the inventory levels may be negative, while this is not happened for the linear demand rates, where the demand rates are linear functions of the inventory levels.

3.3 Logistic demand rates

In this subsection, we will present the model with logistic demand rates. Substituting in the controlled system (2.19) by $D_i(x_1, x_2, t) = x_i(g_i - x_i)$, $\Theta_i = \theta_i x_i$ and $\Theta_i = \theta_i sin(t)$, we get the results in table (4) as following:

Natural deterioration rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J(x_i^*(T), u_i^*(T))$
$\Theta_i = \theta_i x_i$	0.89	0.36	9	8	19.69
$\Theta_i = \theta_i sin(t)$	0.90	0.36	9	8	19.68

Table (4)

3.4 Periodic demand rates

In this subsection, we will present the model with periodic demand rates. Substituting in the controlled system (2.19) by $D_i(x_1, x_2, t) = 1 + k_i \sin t$, $\Theta_i = \theta_i x_i$ and $\Theta_i = \theta_i sin(t)$, we get the results in table (5) as following:

Natural deterioration rates	$x_1^*(T)$	$x_2^*(T)$	$u_1^*(T)$	$u_2^*(T)$	$J(x_i^*(T), u_i^*(T))$
$\Theta_i = \theta_i x_i$	4.19	2.96	9	8	7.60
$\Theta_i = \theta_i sin(t)$	4.26	3.04	9	8	7.65

Table (5)

From previous discussion and results from the tables (2) to (5) which obtained from the numerical solution for the systems (2.19) with different types of the demand and the natural deterioration rates, we have found that: the optimal inventory levels start from initial values and then increase to tend to their goal levels as t increases. The optimal production rates tend to their goal rates as t increases. The aim of this paper is minimizing the total cost and by comparing the total cost in the four types of the demand and deterioration rates, we have found that the constant demand and linear deterioration rates have achieved the minimum value of the total cost.

4 Conclusions

The problem of the optimal control of two-item inventory model with deteriorating items is studied. In this study the inventory model is time-continuous model. The optimal inventory levels and the optimal production rates are derived. Also, the economic conditions that ensure the inventories have no shortage are discussed. The effectiveness of the demand coefficient of one item due to presence of unit of the other is discussed. The behavior of the total cost is investigated and how much affected by the demand and deterioration rates. Finally, the total costs increase when the natural deterioration rates are functions of the inventory levels and as functions of the time.

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